DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCES- MATHEMATICS

SEMESTER -III

PARTIAL DIFFERENTIAL EQUATIONS

DEMATH3OLEC3

BLOCK-1

UNIVERSITY OF NORTH BENGAL

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FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

PARTIAL DIFFERENTIAL EQUATIONS

BLOCK-1

BLOCK-2

BLOCK-1 PARTIAL DIFFERENTIAL EQUATIONS

Introdcution to Block

Partial Differential Equations play an important role in modern mathematics.

This book provides an introduction to the basic properties of Partial Differential Equations and to the techniques that have proved useful in analyzing them.

Introduced some advanced concepts. These concepts are useful for research in modern science. Provided most important proofs and solved examples.

In this block we will learn and understand about first and second order partial differential equations, Waves and diffusions, Boundary problems, Harmonic functions and Green’s identities and Green’s functions.

UNIT-1 PRELIMINARIES

STRUTURE

1.0 OBJECTIVE

After studying this unit**,** you should be able to:

Understand about Partial differential equations, Equations of first order, Quasilinear equations, Initial value problems of Cauchy, Non linear equations in two variables.

1.1 INTRODUCTION

After thinking about the meaning of a partial differential equation, we will flex our mathematical muscles by solving a few of them. Then we will see how naturally they arise in the physical sciences. The physics will motivate the formulation of boundary conditions and initial conditions.

1.2 PARTIAL DIFFERENTIAL EQUATION

The key defining property of a partial differential equation (PDE) is that there is more than one independent variable x, y.......

There is a dependent variable that is an unknown function of these variables

u(x, y, ...). We will often denote its derivatives by subscripts; thus $\partial u/\partial x$ u_x , and so on. A PDE is an identity that relates the independent variables, the dependent variable u, and the partial derivatives of u. It can be written as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0. \qquad (1)$$

This is the most general PDE in two independent variables of first order. The order of an equation is the highest derivative that appears. The most general second-order PDE in two independent variables is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \ldots\ldots\ldots\ldots(2)$$

A solution of a PDE is a function u(x, y,....................) that satisfies the equation

identically, at least in some region of the x, y, variables.

When solving an ordinary differential equation (ODE), one sometimes reverses the roles of the independent and the dependent variables—for in- stance, for the separable

ODE $\dfrac{du}{dx} = u^3$

For PDEs, the distinction between the independent variables and the dependent variable (the unknown) is always maintained.

Some examples of PDEs (all of which occur in physical theory) are:

1. $u_x + u_y = 0$ (transport)
2. $u_x + yu_y = 0$ (transport)
3. $u_x + uu_y = 0$ (shock wave)
4. $u_{xx} + u_{yy} = 0$ (Laplace's equation)
5. $u_{tt} - u_{xx} + u^3 = 0$ (wave with interaction)

Notes

6. $u_t + uu_x + u_{xxx} = 0$ (dispersive wave)

7. $u_{tt} + u_{xxxx} = 0$ (vibrating bar)

8. $u_t - iu_{xx} = 0 \left(i = \sqrt{-1}\right)$ (quantum mechanics)

Each of these has two independent variables, written either as x and y or as x and t. Examples 1 to 3 have order one; 4, 5, and 8 have order two; 6 has order three; and 7 has order four. Examples 3, 5, and 6 are distinguished from the others in that they are not "linear." We shall now explain this concept.

Linearity means the following. Write the equation in the form $\wp u = 0$, where $\wp$ is an operator. That is, if v is any function, $\wp v$ is a new function. For instance, $\wp = \partial/\partial x$ is the operator that takes v into its partial derivative v_x. In Example 2, the operator $\wp$ is $\wp = \partial/\partial x + y\partial/\partial y$. ($\wp u = u_x + yu_y$.) The definition we want for linearity is

$\wp(u + v) = \wp u + \wp v \quad \wp(cu) = c\wp u$ (3)

for any functions u, v and any constant c. Whenever (3) holds (for all choices of u, v, and c), $\wp$ is called linear operator. The equation

$\wp u = 0$

(4)

is called linear if l is a linear operator. Equation (4) is called a homogeneous linear equation. The equation

$\wp u = g$ (5)

/

where $g \neq 0$ is a given function of the independent variables, is called an

inhomogeneous linear equation. For instance, the equation

$(\cos xy^2)u_x - y^2u_y = \tan(x^2 + y^2)$ (6)

is an inhomogeneous linear equation.

As you can easily verify, five of the eight equations above are linear as well as homogeneous. Example 5, on the other hand, is not linear because although

$(u + v)_{xx} = u_{xx} + v_{xx}$ and $(u + v)_{tt} = u_{tt} + v_{tt}$ satisfy property (3), the cubic term does not:

$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 \neq u^3 + v^3$.

The advantage of linearity for the equation $\wp u = 0$ is that if u and v are both solutions, so is (u v). If $u_1, \ldots, u_n$ are all solutions, so is any linear combination

$$c_1u_1(x)+\cdots+c_nu_n(x) = \quad c_ju_j(x) \qquad (c_j = \text{constants}).$$

(This is sometimes called the superposition principle.) Another consequence of linearity is that if you add a homogeneous solution [a solution of (4)] to an inhomogeneous solution [a solution of (5)], you get an inhomogeneous solution. (Why?)

The mathematical structure that deals with linear combinations and linear operators is the vector space.

We'll study, almost exclusively, linear systems with constant coefficients. Recall that for ODEs you get linear combinations. The coefficients are the arbitrary constants. For an ODE of order m, you get m arbitrary constants.

Let's look at some PDEs.

Example 1:

Find all u(x, y) satisfying the equation $u_{xx=}$ 0. Well, we can integrate once to get

u_x =constant. But that's not really right since there's another variable y.

What we really get is $u_x(x, y)= f(y)$, where f (y) is arbitrary.

Do it again to get $u(x, y)= f(y)x+ g(y)$.

This is the solution formula. Note that there are two arbitrary functions in the solution.

Example 2:

Solve the PDE $u_{xx} +u= 0$. Again, it's really an ODE with an extra variable y. We know how to solve the ODE, so the solution is

$u = f(y) \cos x + g(y) \sin x$,

where again f (y) and g(y) are two arbitrary functions of y.

You can easily check this formula by differentiating twice to verify that $u_{xx} = -u$.

Example 3:

Solve the PDE $u_{xy=}0$. This isn't too hard either. First let's integrate in x, regarding y as fixed. So we get $u_y(x, y) = f(y)$.

Next let's integrate in y regarding x as fixed. We get the solution.

$u(x, y) = F(y) + G(x)$,

where $F^1 = f$.

Note: A PDE has arbitrary functions in its solution. In these examples the arbitrary functions are functions of one variable that combine to produce a function u(x, y) of two variables which is only partly arbitrary.

A function of two variables contains immensely more information than a function of only one variable. Geometrically, it is obvious that a surface $u= f(x, y)$, the graph of a function of two variables, is a much more complicated object than a curve $u=f(x)$, the graph of a function of one variable.

To illustrate this, we can ask how a computer would record a function u f (x). Suppose that we choose 100 points to describe it using equally spaced values of x: x_1, x_2,

$x_3, \ldots, x_{100}$.

We could write them down in a column, and next to each x_j we could write the

corresponding value $u_{j=} f(x_j)$.

Now how about a function u f (x, y)? Suppose that we choose 100 equally spaced values of x and also of y: $x_1, x_2, x_3, \ldots, x_{100}$ and $y_1, y_2, y_3, \ldots, y_{100}$.

Each pair x_i, y_j provides a value $u_{ij=} f(x_i, y_j)$, so there will be 100^2 = 10,000 lines of the form $x_i \quad y_j \quad u_{ij}$

required to describe the function! (If we had a prearranged system, we would need to record only the values u_{ij}.) A function of three variables described discretely by 100 values in each variable would require a million numbers!

To understand this book what do you have to know from calculus? Certainly all the basic facts about partial derivatives and multiple integrals. For a brief discussion of such topics, see the Appendix. Here are a few things to keep in mind, some of which may be new to you.

1. Derivatives are local. For instance, to calculate the derivative $(\partial u/\partial x)(x_0, t_0)$ at a particular point, you need to know just the values of u(x,

t_0) for x near x_0, since the derivative is the limit as $x \to x_0$.

2. Mixed derivatives are equal: u_{xy} u_{yx}. (We assume throughout this book, unless stated otherwise, that all derivatives exist and are continuous.)
3. The chain rule is used frequently in PDEs; for instance,

$$\frac{\partial}{\partial x}[f (g(x, t))]=f^{1}(g(x, t)) \frac{\partial g}{\partial x} (x, t).$$

4. Derivatives of integrals like $I (t) = \int_{a(t)}^{b(t)} f (x, t) dx$

Exercise:

1. Verify the linearity and nonlinearity of the eight examples of PDEs given in the text, by checking whether or not equations (3) are valid.

2. Which of the following operators are linear?

(a) $\wp u = u_x + x u_y$

(b) $\wp u = u_x + u u_y$

(c) $\wp u = u_x + u^2$

(d) $\wp u = u_x + u_y + 1$

3. Are the functions 1+x, 1-x and $1+x+x^2$ linearly dependent or independent? Why?

1.3 EQUATIONS OF FIRST ORDER

For a given sufficiently regular function F the general equation of first order for the unknown function u(x) is $F (x, u, \nabla u) = 0$

in $\Omega \in R^n$. The main tool for studying related problems is the theory of ordinary differential equations. This is quite different for systems of partial differential of first order.

The general linear partial differential equation of first order can be written as

$$\sum_{i=1}^{n} ai(x)uxi + c(x)u = f (x)$$ for given functions a_i, c and **f** .

The general quasilinear partial differential equation of first order is

$$\sum_{i=1}^{n} ai(x,u)uxi + c(x,u) = 0$$

Notes

Check your progress

1. Discuss about equation of first order

1.4 LINEAR EQUATIONS

Let us begin with the linear homogeneous equation

$$a_1(x, y)u_x + a_2(x, y)u_y = 0. \qquad (1)$$

Assume there is a C^1-solution z = u(x, y). This function defines a surface

S which has at P = (x, y, u(x, y)) the normal

$N = \dfrac{1}{\sqrt{1+\nabla u^2}}(-u_x, -u_y, 1)$ and the tangential plane defined by

$\zeta - z = u_x(x, y)(\xi - x) + u_y(x, y)(\eta - y)$.

Set $p = u_x(x, y)$, $q = u_y(x, y)$ and $z = u(x, y)$.

The tuple (x, y, z, p, q) is called surface element and the tuple (x, y, z) support of the surface element. The tangential plane is defined by the surface element.

On the other hand, differential equation (1)

$a_1(x, y)p + a_2(x, y)q = 0$

defines at each support (x, y, z) a bundle of planes if we consider all (p, q) satisfying this equation.

For fixed (x, y), this family of planes $\Pi(\lambda) = \Pi(\lambda; x, y)$ is defined by a one parameter family of ascents $p(\lambda) = p(\lambda; x, y)$, $q(\lambda) = q(\lambda; x, y)$.

The envelope of these planes is a line since

$a_1(x, y)p(\lambda) + a_2(x, y)q(\lambda) = 0$,

which implies that the normal $N(\lambda)$ on $\Pi(\lambda)$ is perpendicular on $(a_1, a_2, 0)$.

Consider a curve $x(\tau) = (x(\tau), y(\tau), z(\tau))$ on S,

Let T_{x_0} be the tangential plane at $x_0 = (x(\tau_0), y(\tau_0), z(\tau_0))$ of S and consider on

T_x= the line

L : $l(\sigma) = x_0 + \sigma x^J(\tau_0), \quad \sigma \in R,$

By figure (1),

We assume L coincides with the envelope, which is a line here, of the family of planes $\Pi(\lambda)$ at (x, y, z). Assume that $T_{x0} = \Pi(\lambda_0)$ and consider two planes

$\Pi(\lambda_0)$: $z - z_0 = (x - x_0)p(\lambda_0) + (y - y_0)q(\lambda_0)$

$\Pi(\lambda_0 + h)$: $z - z_0 = (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h).$

At the intersection $l(\sigma)$ we have

$(x - x_0)p(\lambda_0) + (y - y_0)q(\lambda_0) = (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h).$

Thus,

$x^J(\tau_0)p^J(\lambda_0) + y^J(\tau_0)q^J(\lambda_0) = 0.$

From the differential equation

$a_1(x(\tau_0), y(\tau_0))p(\lambda) + a_2(x(\tau_0), y(\tau_0))q(\lambda) = 0$

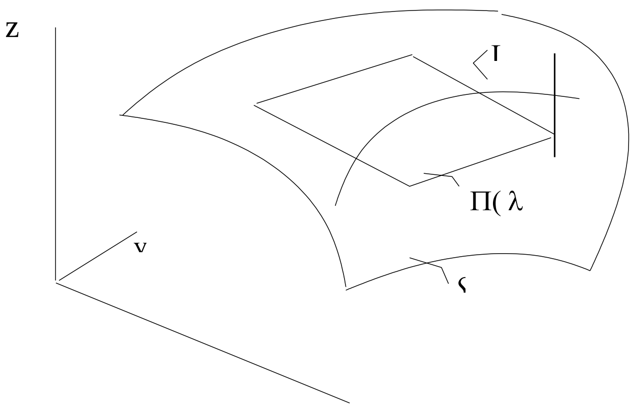


Figure (1)

Figure 1: Curve on a surface

it follows $a_1 p^1(\lambda_0) + a_2 q^1(\lambda_0) = 0$

.Consequently,

$$(x^1(\tau), y^1(\tau)) = \frac{x^1(\tau)}{a^1(x(\tau, y(\tau))}(a\ (x(\tau), y(\tau)), a\ (x(\tau), y(\tau)),$$

since τ_0 was an arbitrary parameter. Here we assume that $x^J(\tau)$ *f*= 0 and

$a_1(x(\tau), y(\tau))$ *f* ≠ 0.

Then we introduce a new parameter t by the inverse of $\tau = \tau(t)$, where

$$\int_{\tau_0}^{\tau} \frac{x^1(s)}{a_1(x(s), y(s))} ds$$

It follows $x'(t) = a_1(x, y)$, $y'(t) = a_2(x, y)$. We denote $x(\tau(t))$ by $x(t)$ again.

Now we consider the initial value problem

$$x'(t) = a_1(x, y), \quad y'(t) = a_2(x, y), \quad x(0) = x_0, \; y(0) = y_0. \tag{2}$$

From the theory of ordinary differential equations it follows (Theorem of Picard-Lindelöf) that there is a unique solution in a neighborhood of $t = 0$ provided the functions a_1, a_2 are in C^1.

From this definition of the curves

$(x(t), y(t))$ is follows that the field of directions $(a_1(x_0, y_0), a_2(x_0, y_0))$ defines the slope of these curves at $(x(0), y(0))$.

Definition: The differential equations in (2) are called characteristic equations or characteristic system and solutions of the associated initial value problem are called characteristic curves.

Definition: A function $\varphi(x, y)$ is said to be an integral of the characteristic system if $\varphi(x(t), y(t)) = \text{const.}$ for each characteristic curve. The constant depends on the characteristic curve considered.

Proposition 1: Assume $\varphi \in C^1$ is an integral, then $u = \varphi(x, y)$ is a solution of (2.1).

Proof. Consider for given (x_0, y_0) the above initial value problem (2). Since $\varphi(x(t), y(t)) = \text{const.}$ it follows

$\varphi_x x' + \varphi_y y' = 0$ for $|t| < t_0$, $t_0 > 0$ and sufficiently small.

Thus

$\varphi_x(x_0, y_0)a_1(x_0, y_0) + \varphi_y(x_0, y_0)a_2(x_0, y_0) = 0.$

Remark. If $\varphi(x, y)$ is a solution of equation (1) then also $H(\varphi(x, y))$, where $H(s)$ is a given C^1-function.

Examples:

1. Consider

$a_1u_x + a_2u_y = 0,$

where a_1, a_2 are constants. The system of characteristic equations is

$x' = a_1, \; y' = a_2.$

Thus the characteristic curves are parallel straight lines defined by

$x = a_1t + A, \; y = a_2t + B,$

where A, B are arbitrary constants. From these equations it follows that

$\varphi(x, y) := a_2x - a_1y$

is constant along each characteristic curve. Consequently, see Proposition (1),

$u = a_2x - a_1y$ is a solution of the differential equation. From an exercise it follows that

$u = H(a_2x - a_1y)$, (3)

Where H(s) is an arbitrary C^1-function, is also a solution. Since u is constant when $a_2x - a_1y$ is constant, equation (3) defines cylinder surfaces which are generated by parallel straight lines which are parallel to the (x, y)-plane, see Figure 1.2.

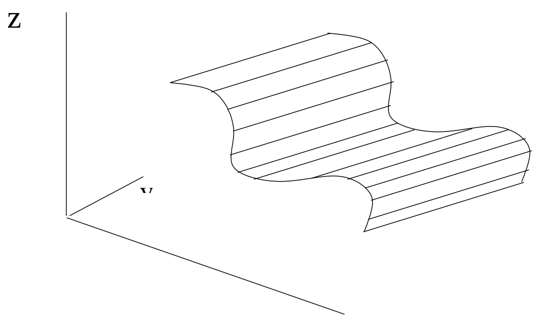


Figure 1.2: Cylinder surfaces

2. Consider the differential equation

$xu_x + yu_y = 0.$

The characteristic equations are

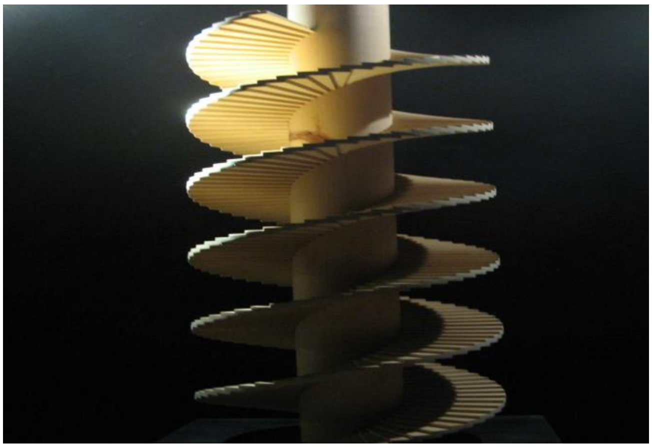
$x' = x$, $y' = y$, nand the characteristic curves are given by

$x = Ae^t$, $y = Be^t$,

where A, B are arbitrary constants.

Thus, an integral is y/x, $x = 0$, and for a given C^1-function the function $u = H(x/y)$ is a solution of the differential equation.

Notes



If y/x = const., then u is constant. Suppose that $H^1(s) > 0$, for example, then u defines right helicoids (in German: Wendelflächen), see Figure 1.3

Figure 1.3: Right helicoid, $a^2 < x^2 + y^2 < R^2$ (Museo Ideale Leonardo da Vinci, Italy)

3. Consider the differential equation

$yu_x - xu_y = 0.$

The associated characteristic system is

$x^1 = y,\ \ y^1 = -x$.If follows

$x^1x + yy^1 = 0$, or equivalently, $\frac{d}{dt}(x^2 + y^2) = 0$

which implies that $x^2 + y^2$ = const. along each characteristic. Thus, rotationally symmetric surfaces defined by $u = H(x^2 + y^2)$, where H^J $f= 0$, are solutions of the differential equation.

4. The associated characteristic equations to

$ayu_x + bxu_y = 0$, where a, b are positive constants, are given by

$x^1 = ay,\ \ y^1 = bx.$

It follows $bxx^1 - ayy^1 = 0$, or equivalently,

$\frac{d}{dt}(bx^2 - ay^2) = 0$

Solutions of the differential equation are $u = H(bx^2 - ay^2)$, which define surfaces which have a hyperbola as the intersection with planes parallel to the (x, y)-plane.

Here $H(s)$ is an arbitrary C^1-function, $H^1(s)$ $f= 0$.

1.5 QUASILINEAR EQUATIONS

Here we consider the equation

$a_1(x, y, u)u_x + a_2(x, y, u)u_y = a_3(x, y, u).$ (4)

The inhomogeneous linear equation

$a_1(x, y)u_x + a_2(x, y)u_y = a_3(x, y)$ is a special case of (4).

One arrives at characteristic equations $x^1 = a_1$, $y^1 = a_2$, $z^1 = a_3$ from (4) by the same arguments as in the case of homogeneous linear equations in two variables. The additional equation $z^1 = a_3$ follows from

$Z^1(\tau) \quad = \quad p(\lambda)1^1(\tau) + q(\lambda)y^1(\tau)$

$= pa_1 + qa_2 = a_3,$

A linearization method

We can transform the inhomogeneous equation (4) into a homogeneous linear equation for an unknown function of three variables by the following trick.

We are looking for a function $\psi(x, y, u)$ such that the solution $u = u(x, y)$ of (4) is defined implicitly by $\psi(x, y, u) = \text{const}$.

Assume there is such a function ψ and let u be a solution of (4), then

$\psi_x + \psi_u u_x = 0, \ \psi_y + \psi_u u_y = 0.$

Assume $\psi_u \neq 0$, then

$$ux = -\frac{\psi x}{\psi u}, \quad uy = -\frac{\psi y}{\psi u}$$

From (4)

$a_1(x, y, z)\,\psi_x + a_2(x, y, z)\psi_y + a_3(x, y, z)\psi_z = 0,$ (5)

where $z := u$.

We consider the associated system of characteristic equations

$$x^1(t) = a_1(x, y, z)$$
$$y^1(t) = a_2(x, y, z)$$
$$z^1(t) = a_3(x, y, z)$$

One arrives at this system by the same arguments as in the two-dimensional case above.

Proposition 2:

Notes

(i) Assume $w \in C^1$, $w = w(x, y, z)$, is an integral, i. e., it is constant along each fixed solution of (5), then $\psi = w(x, y, z)$ is a solution of (5).

(ii) The function $z = u(x, y)$, implicitly defined through $\psi(x, u, z) =$ const., is a solution of (4), provided that $\psi_z \neq 0$.

(iii) Let $z = u(x, y)$ be a solution of (4) and let $(x(t), y(t))$ be a solution of

$$x'(t) = a_1(x, y, u(x, y)), \quad y'(t) = a_2(x, y, u(x, y)),$$

then $z(t) := u(x(t), y(t))$ satisfies the third of the above characteristic equations.

Check your progress

2. Discuss about quasilinear equations

1.6 INITIAL VALUE PROBLEM OF CAUCHY

Consider again the quasilinear equation

$$(*) a_1(x, y, u)u_x + a_2(x, y, u)u_y = a_3(x, y, u).$$

Let $\Gamma : x = x_0(s),\ y = y_0(s),\ z = z_0(s),\ s_1 \leq s \leq s_2,\ -\infty < s_1 < s_2 < +\infty$

be a regular curve in R^3 and denote by C the orthogonal projection of Γ onto the

(x, y)-plane, i. e.,

$$C : \quad x = x_0(s), \quad y = y_0(s).$$

Initial value problem of Cauchy: Find a C^1-solution $u = u(x, y)$ of $(\times)$ such that $u(x_0(s), y_0(s)) = z_0(s)$, i. e., we seek a surface S defined by $z = u(x, y)$ which contains the curve Γ.

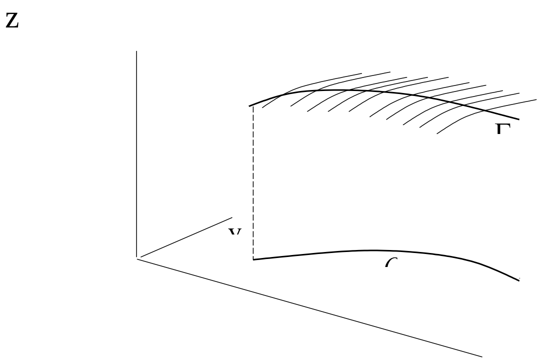


Figure 1.4: Cauchy initial value problem

Definition: The curve Γ is said to be non characteristic if

$x^1{}_0(s)a_2(x_0(s), y_0(s)) - y^1{}_0(s)a^1(x_0(s), y_0(s)) \neq 0$.

Theorem: Assume a_1, a_2, $a_2 \in C^1$ in their arguments, the initial data x_0, y_0, $z_0 \in C^1[s_1, s_2]$ and Γ is non characteristic.

Then there is a neighborhood of C such that there exists exactly one solution u of the Cauchy initial value problem.

Proof. (i) Existence. Consider the following initial value problem for the system of characteristic equations to (*):

$a_1(x, y, z)$
$y^1(t)=a_2(x, y, z)$
$z^1(t)=a_3(x, y, z)$
with the initial conditions
$x(s,0)=x_0(s)$
$y(s,0)=y_0(s)$
$z(s,0)=z_0(s)$

Let $x = x(s, t)$, $y = y(s, t)$, $z = z(s, t)$ be the solution, $s_1 \leq s \leq s_2$, $|t| < \eta$ for an $\eta > 0$. We will show that this set of strings stickled onto the curve Γ, see Figure 1.4, defines a surface. To show this, we consider the inverse functions $s = s(x, y)$, $t = t(x, y)$ of $x = x(s, t)$, $y = y(s, t)$ and show that $z(s(x, y), t(x, y))$ is a solution of the initial problem of Cauchy.

The inverse functions s and t exist in a neighborhood of $t = 0$ since

$$\det \frac{\partial(x,y)}{\partial(s,t)\ t=0} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix}_{t=0} = x^1{}_0(s)a_2 - y^1{}_0(s)a_1 \neq 0$$

and the initial curve Γ is non characteristic by assumption.

Set $u(x, y) := z(s(x, y), t(x, y))$,

then u satisfies the initial condition since $u(x, y)|_{t=0} = z(s, 0) = z_0(s)$.

The following calculation shows that u is also a solution of the differential equation (×).

$$\begin{aligned} a_1u_x + a_2u_y &= a_1(z_ss_x + z_tt_x) + a_2(z_ss_y + z_tt_y) \\ &= z_s(a_1s_x + a_2s_y) + z_t(a_1t_x + a_2t_y) \\ &= z_s(s_xx_t + s_yy_t) + z_t(t_xx_t + t_yy_t) \\ &= a_3 \end{aligned}$$

since $0 = s_t = s_x x_t + s_y y_t$ and $1 = t_t = t_x x_t + t_y y_t$.

(ii) Uniqueness. Suppose that $v(x, y)$ is a second solution. Consider a point (x^J, y^J) in a neighborhood of the curve $(x_0(s), y(s))$,

$s_1\text{-}s \le s \le s_2 + s, s > 0$ small.

(iii) The inverse parameters are $s^J = s(x^J, y^J)$, $t^J = t(x^J, y^J)$, see Figure 1.5.

Figure 1.5: Uniqueness proof Let

$A : x(t) := x(s1, t)$, $y(t) := y(s1, t)$, $z(t) := z(s1, t)$

be the solution of the above initial value problem for the characteristic differential equations with the initial data

$x(s^1, 0) = x_0(s^1)$, $y(s^1, 0) = y_0(s^1)$, $z(s^1, 0) = z_0(s^1)$.

According to its construction this curve is on the defined by

$u = u(x, y)$ and $u(x^1, y^1) = z(s^1, t^1)$. Set

$\psi(t) := v(x(t), y(t)) - z(t)$,

Then $\Psi^1(t) = vxx^1 + vyy^J = z^1$

$= xxa1 + vya2 - a3 = 0$

$\psi(0) = v(x(sj, 0), y(sj, 0)) - z(sj, 0) = 0$

since v is a solution of the differential equation and satisfies the initial con- dition by assumption. Thus, $\psi(t) \equiv 0$, i. e.,

$v(x(s^J, t), y(s^J, t)) - z(s^J, t) = 0$.

Set $t = t^J$, then

$v(xj, yj) - z(sj, tj) = 0$,which shows that $v(x^J, y^J) = u(x^J, y^J)$ because of $z(s^J, t^J) = u(x^J, y^J)$.

Remark. In general, there is no uniqueness if the initial curve Γ is a characteristic curve, see an exercise and Figure 1.6 which illustrates this case.

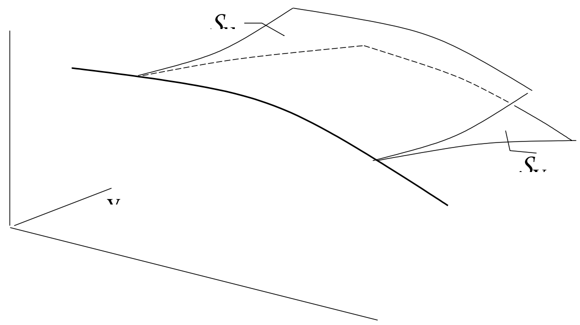


Figure 1.6: Multiple solutions

Examples:

1. Consider the Cauchy initial value problem

$u_x + u_y = 0$ with the initial data

$x_0(s) = s$, $y_0(s) = 1$, $z_0(s)$ is a given C^1-function.

These initial data are non characteristic since $y_0' a_1 - x_0' a_2 = -1$. The solution of the associated system of characteristic equations

$x'(t) = 1, \; y'(t) = 1, \; u'(t) = 0$

with the initial conditions

$x(s, 0) = x_0(s)$, $y(s, 0) = y_0(s)$, $z(s, 0) = z_0(s)$ is given by

$x = t + x0(s)$, $y = t + y0(s)$, $z = z0(s)$, $x = t + s$, $y = t + 1$, $z = z0(s)$.

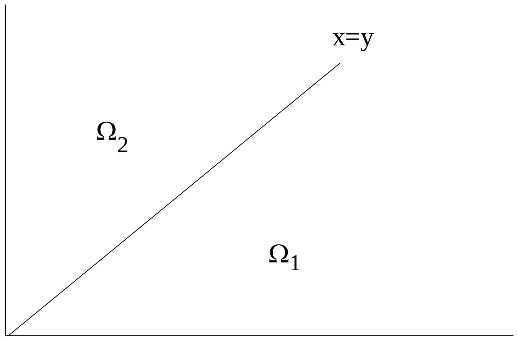
It follows $s = x - y + 1$, $t = y - 1$ and that $u = z_0(x - y + 1)$ is the solution of the Cauchy initial value problem.

A problem from kinetics in chemistry. Consider for $x \geq 0$, $y \geq 0$ the problem with initial data.

$u(x, 0) = 0$, $x > 0$, and $u(0, y) = u_0(y)$, $y > 0$.

Here the constants k_j are positive, these constants define the velocity of the reactions in consideration, and the function $u_0(y)$ is given. The variable x is the time and y is the height of a tube, for example, in which the chemical reaction takes place, and u is the concentration of the chemical substance.

In contrast to our previous assumptions, the initial data are not in C^1. The projection $C_1 \cup C_2$ of the initial curve onto the (x, y)-plane has a corner at the origin, see Figure 1.7.



The associated system of characteristic equations is

$$x'(t) = 1,\ \ y'(t) = 1,\ \ z'(t) = \left(k_0 e^{-k_1 x} + k_2\right)(1 - z).$$

It follows $x = t + c_1$, $y = t + c_2$ with constants c_j.

Thus the projection of the characteristic curves on the (x, y)-plane are straight lines parallel to $y = x$. We will solve the initial value problems in the domains Ω_1 and Ω_2, see Figure 2.7, separately.

(i) The initial value problem in Ω_1. The initial data are

$x_0(s) = s,\ y_0(s) = 0,\ z_0(0) = 0,\ s \geq 0.$

It follows $x = x(s, t) = t + s$, $y = y(s, t) = t$.

$$z'(t) = (k_0 e^{-k_1(t+s)} + k_2)(1 - z),\ \ z(0) = 0$$

The solution of this initial value problem is given by

$$z(s, t) = 1 - \exp\left(\frac{k_0}{k_1}\, e^{-k_1(s+t)} - k_2 t - \frac{k_0}{k_1}\, e^{-k_1 s}\right).$$

Consequently

$$u(x, y) = 1 - \exp\left(\frac{k_0}{k_1}\, e^{-k_1 x} - k_2 y - k_0 k_1\, e^{-k_1(x-y)}\right)$$

is the solution of the Cauchy initial value problem in Ω_1. If time x tends to

∞, we get the limit. $\lim\limits_{x \to \infty} u_1(x,y) = 1 - e^{-k^2 y}$

(ii) The initial value problem in Ω_2. The initial data are here

$x_0(s) = 0,\ y_0(s) = s,\ z_0(0) = u_0(s),\ s \geq 0.$

It follows

$x = x(s, t) = t$, $y = y(s, t) = t + s$.

$$z^1(t) = (k_0 e^{-k_1 t} + k_2)(1 - z),\ \ z(0) = 0.$$

The solution of this initial value problem is given by

$$z(s, t) = 1 - (1-u_0(s)\exp\left(\frac{k_0}{k_1}\,e^{-k_1 t} - k_2 t - \frac{k_0}{k_1}\right)$$

Consequently,

$$u_2(x, y) = 1 - (1-u_0(y-x)\exp\left(\frac{k_0}{k_1}\,e^{-k_1 x} - k_2 x - \frac{k_0}{k_1}\right)$$

Is the solution in Ω_2

If x=y then

$$u_1(x, y) = 1 - \exp\left(\frac{k_0}{k_1}\,e^{-k_1 x} - k_2 x - \frac{k_0}{k_1}\right)$$

$$u_2(x, y) = 1 - (1-u_0(0)\exp\left(\frac{k_0}{k_1}\,e^{-k_1 x} - k_2 x - \frac{k_0}{k_1}\right)$$

If $u_0(0) > 0$, then $u_1 < u_2$ if $x = y$, i. e., there is a jump of the concentration of the substrate along its burning front defined by $x = y$.

Remark. Such a problem with discontinuous initial data is called Riemann problem. See an exercise for another Riemann problem.

The case that a solution of the equation is known

Here we will see that we get immediately a solution of the Cauchy initial value problem if a solution of the homogeneous linear equation

$a_1(x, y)u_x + a_2(x, y)u_y = 0$ is known.

Let $x_0(s)$, $y_0(s)$, $z_0(s)$, $s_1 < s < s_2$

be the initial data and let $u = \varphi(x, y)$ be a solution of the differential equation.

We assume that

$$\varphi_x(x_0(s), y_0(s))x_0'(s) + \varphi_y(x_0(s), y_0(s))y_0'(s) \neq 0$$

is satisfied. Set $g(s) = \varphi(x_0(s), y_0(s))$ and let $s = h(g)$ be the inverse function.

The solution of the Cauchy initial problem is given by $u_0(h(\varphi(x, y)))$.

This follows since in the problem considered a composition of a solution is a solution again, see an exercise, and since

$u_0(h(\varphi(x_0(s), y_0(s))) = u_0(h(g)) = u_0(s)$.

Example: Consider equation $u_x + u_y = 0$

With initial data $x_0(s) = s$, $y_0(s) = 1$, $u_0(s)$ is a given function.

A solution of the differential equation is $\varphi(x, y) = x - y$. Thus

Notes

$\varphi((x_0(s), y_0(s)) = s - 1$

$u_0(\phi+1) = u_0(x - y + 1)$

is the solution of the problem.

1.7 NON LINEAR EQUATIONS IN TWO VARIABLES

Here we consider equation

$$F(x, y, z, p, q) = 0, \qquad (6)$$

where $z = u(x, y)$, $p = u_x(x, y)$, $q = u_y(x, y)$ and $F \in C^2$ is given such that

$$F^2_p + F^2_q \neq 0$$

In contrast to the quasilinear case, this general nonlinear equation is more complicated.

Together with (6) we will consider the following system of ordinary equations which follow from considerations below as necessary conditions, in particular from the assumption that there is a solution of (6).

$$x^1(t) = F_p \qquad (7)$$

$$y^1(t) = F_q \qquad (8)$$

$$z^1(t) = pF_p + qF_q \qquad (9)$$

$$p^1(t) = -F_x - F_u p \qquad (10)$$

$$q^1(t) = -F_y - F_u q. \qquad (11)$$

We will see, as in the quasilinear case, that the strips defined by the characteristic equations build the solution surface of the Cauchy initial value problem.

Let $z = u(x, y)$ be a solution of the general nonlinear differential equation (6).

Let (x_0, y_0, z_0) be fixed, then equation (6) defines a set of planes given by (x_0, y_0, z_0, p, q), i. e., planes given by $z = v(x, y)$ which contain the point (x_0, y_0, z_0) and for which $v_x = p$, $v_y = q$ at (x_0, y_0).

Theorem : There exists a solution of the Cauchy initial value problem provided the initial data are non characteristic. That is, we do not need the other remaining two characteristic equations.

The other two equations (10) and (11) are satisfied in this quasilinear case automatically if there is a solution of the equation, see the above derivation of these equations.

The geometric meaning of the first three characteristic differential equations (7)–(11) is the following one.

Each point of the curve

A : (x(t), y(t), z(t)) corresponds a tangential plane with the normal direction (−p, −q, 1)

such that

$z^1(t) = p(t)x^1(t) + q(t)y^1(t)$.

This equation is called strip condition.

On the other hand, let z = u(x, y) defines a surface, then z(t) := u(x(t), y(t)) satisfies the strip condition, where $p = u_x$ and $q = u_y$, that is, the ”scales” defined by the normals fit together.

Proposition 3: F (x, y, z, p, q) is an integral, i. e., it is constant along each characteristic curve.

Proof:

$$\frac{\mathrm{d}}{\mathrm{dt}}F(x(t),y(t),z(t),p(t),q(t) = F_x x^1 + F_y y^1 + F_z z^1 + F_p p^1 + F_q q^1$$
$$= F_x F_p + F_y F_q + pF_z F_p + qF_z F_q$$
$$- F_q F p - F_p F_z p - F_q F_y - F_q F_z q$$
$$= 0$$

Corollary. Assume $F(x_0, y_0, z_0, p_0, q_0) = 0$, then F = 0 along characteristic curves with the initial data $(x_0, y_0, z_0, p_0, q_0)$.

Proposition 4:

Let z = u(x, y), $u \in C^2$, be a solution of the nonlinear equation (2.6). Set

$z_0 = u(x_0, y_0,)$ $p_0 = u_x(x_0, y_0)$, $q_0 = u_y(x_0, y_0)$.

Then the associated characteristic strip is in the surface S, defined by z =

u(x, y). Thus

$$z(t) = u(x,(t), y(t)$$
$$p(t) = u_x(x,(t), y(t)$$
$$q(t) = u_y(x,(t), y(t)$$

Notes

where $(x(t), y(t), z(t), p(t), q(t))$ is the solution of the characteristic system (7)–(11) with initial data $(x_0, y_0, z_0, p_0, q_0)$

Proof: Consider the initial value problem

$x^1(t) \quad = \quad F_p(x, y, u(x, y), u_x(x, y), u_y(x, y))$

$y^1(t) \quad = \quad F_q(x, y, u(x, y), u_x(x, y), u_y(x, y))$ with the initial data $x(0) = x_0$, $y(0) = y_0$. We will show that

$(x(t), y(t), u(x(t), y(t)), u_x(x(t), y(t)), u_y(x(t), y(t)))$

is a solution of the characteristic system. We recall that the solution exists and is uniquely determined.

Set $z(t) = u(x(t), y(t))$, then $(x(t), y(t), z(t)) \subset S$, and

$z^1(t) = u_x x^1(t) + u_y y^1(t) = u_x F_p + u_y F_q$.

Set $p(t) = u_x(x(t), y(t))$, $q(t) = u_y(x(t), y(t))$, then

$P^1(t) \quad = \quad u_{xx} F_p + u_{xy} F_q \; q^1(t) \quad = \quad u_{yx} F_p + u_{yy} F_q$.

Finally, from the differential equation $F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0$ it follows

$p^J(t) \quad = \quad -F_x - F_u p \; q^J(t) \quad = \quad -F_y - F_u q$.

Definition: A strip $(x(\tau), y(\tau), z(\tau), p(\tau), q(\tau))$, $\tau 1 < \tau < \tau 2$, is said to be Non characteristic if

$x^J(\tau)F_q(x(\tau), y(\tau), z(\tau), p(\tau), q(\tau)) - y^J(\tau)F_p(x(\tau), y(\tau), z(\tau), p(\tau), q(\tau)) \, f\!= 0$.

Example. A differential equation which occurs in the geometrical optic is

$$u_x^2 + u_y^2 = f(x, y),$$

where the positive function $f(x, y)$ is the index of refraction. The level sets defined by $u(x, y) =$ const. are called wave fronts. The characteristic curves $(x(t), y(t))$ are the rays of light. If n is a constant, then the rays of light are straight lines. In R^3 the equation is

$$u_x^2 + u_y^2 + u_z^2 = f(x, y, z).$$

Thus we have to extend the previous theory from R^2 to R^n, $n \geq 3$.

EXERCISE:

1. Solve the initial value problem

$xu_x + yu_y = u$ with initial data $x_0(s) = s$, $y_0(s) = 1$, $z_0(s)$, where z_0 is given.

2. Solve the initial value problem

$-xu_x + yu_y = xu$, $x_0(s) = s$, $y_0(s) = 1$, $z_0(s) = e^{-s}$.

3. Solve the initial value problem

$uu_x + u_y = 1$, $x_0(s) = s$, $y_0(s) = s$, $z_0(s) = s/2$ if $0 < s < 1$.

Check your progress

3. Discuss about non linear equations in two variables

1.8 LET US SUM UP

In this unit we have discussed the definition of partial differential equation, Regular function F the general equation of first order for the unknown function.

Characteristic equations or characteristic system and solutions of the associated initial value problems of Cauchy. Non linear equations in two variables. Solution of the general nonlinear differential equation. Solution of the Cauchy initial value problem provided the initial data are non characteristic.

1.9 KEY WORDS

1. The key defining property of a partial differential equation (PDE) is that there is more than one independent variable x, y…….
2. A PDE is an identity that relates the independent variables, the dependent variable u, and the partial derivatives of u. It can be written as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0.$$

3. The chain rule is used frequently in PDEs; for instance,

$$\frac{\partial}{\partial x}[\, f\,(g(x, t\,))]=f^{\,1}(g(x, t\,))\;\frac{\partial g}{\partial x}(x, t\,).$$

4. The general linear partial differential equation of first order can be written as

$$\sum_{i=1}^{n} ai(x)uxi + c(x)u = f(x) \text{ for given functions } a_i, c \text{ and } f .$$

5. The general quasilinear partial differential equation of first order is

$$\sum_{i=1}^{n} ai(x,u)uxi + c(x,u) = 0$$

6. Cauchy initial value problem if a solution of the homogeneous linear equation
7. F (x, y, z, p, q) is an integral, i. e., it is constant along each characteristic curve.

1.10 QUESTIONS FOR REVIEW

1. Discuss about equation of first order
2. Discuss about quasilinear equations
3. Discuss about non linear equations in two variables

1.11 SUGGESTIVE READINGS AND REFERENCES

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1.12 ANSWERS TO CHECK YOUR PROGRESS

1. See section 1.3
2. Sec section 1.5
3. See section 1.7

UNIT – 2 SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

STRUTURE

2.0 OBJECTIVE

After studying this unit we should able to learn about linear equations of second order, Normal form in two variables, Quasilinear equations of second order, Quasilinear elliptic.

2.1 INTRODUCTION

The classification of differential equations follows from one single question: Can we calculate formally the solution if sufficiently many initial data are given?

Consider the initial problem for an ordinary differential equation $y'(\mathrm{x}) = \mathrm{f}(\mathrm{x}, \mathrm{y}(\mathrm{x})), \mathrm{y}(\mathrm{x}_0) = y_o$. Then one can determine formally the solution , provided the function $f(x, y)$ the initial value problem is formally given by a power series. This formal solution is a solution of the problem if f(x,y) is real analytic according to a theorem of Cauchy.

Even in the case of ordinary differential equations the situation is more complicated if is implicitly defined, i. e., the differential equation is $F(x, y)(\mathrm{x})) = 0$ for a given function F.

2.2 LINEAR EQUATIONS OF SECOND ORDER

The general non linear partial differential equation of second order is

$$F(x, u, Du, D^2 v) = 0.$$

Where $x \in R^n, u : \Omega R^{"} \to R, Du \equiv \nabla_u$ and stands for all second derivatives. The function F is given and sufficiently regular with respect to its $2n + 1 + n^2$ arguments.

In this section we consider the case

$$\sum_{i,k=1}^{n} a^{ik}(x) u_{x_1} x_k + f(x, u, \nabla u) = 0. \text{--------}(2.1)$$

The equation is liner if

$$f = \sum_{i=1}^{n} b^2(x) u_{x_1} + c(x) u + d(x).$$

Concerning the classification the main part

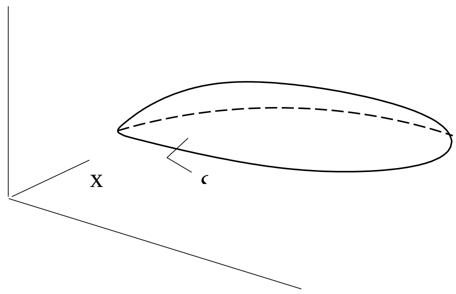
$$\sum_{i,k=1}^{n} a^{ik}(x) u_{x_1 x_k}$$

Plays the essential role. Suppose $u \in C^2$, then we can assume, without restriction of generality, that $a^{ik} = a^{ki,}$, since

$$\sum_{i,k=1}^{n} a^{ik} u_{x_1 x_k} = \sum_{i,k=1}^{n} (a^{ik})^{+} u_{x_1 x_k},$$

Where $(a^{ik})^{*} = \frac{1}{2}(a^{ik} + a^{ki})$.

Consider a hyper surface $S \, in \, R^n$ defined implicitly by $x(x) = 0, \nabla x \neq 0,$ see figure 2.1



Assume ∇u *are given on S*.

Problem: Can we calculate all other derivatives of a an S by using differential equal (3.1) and the given data ?

We will find an answer if we map S onto a hyper plane $u \; on S$ by a mapping

$$\lambda_n = x(x_1, \ldots\ldots, x_n)$$

$$\lambda_i = \lambda_i(x_1, \ldots\ldots, x_n), i = 1, \ldots, n-1,$$

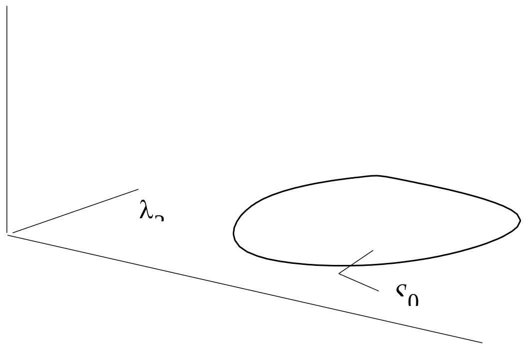
For functions λ_i such that

$$\det \frac{\partial(\lambda_1, \ldots\ldots \lambda_n}{\partial(x_1, \ldots . x_n)} \neq 0$$

In $\Omega \subset R^n$. It is assumed that $\chi \; and \; \lambda_i$ are sufficiently regular. Such mapping $\lambda = \lambda(x)$ exists, see an exercise.

Figure 2.1: Initial manifold S

The above transform maps S onto a subset of the hyper plane defined by $\lambda_n = 0,$



We will write the differential equation in these new coordinates. Here we use Einstein's convention i.e., we add terms with repeating indices. Since

$$u(x) = u(x(\lambda)) =; v(\lambda(x)),$$

Where $x = (x_1, ..., x_n) and\ \lambda = (\lambda_1, ..., \lambda_n),$ we get

$$u_{x_3 =} v_{\lambda_i} \frac{\partial \lambda_i}{\partial x_j}, \text{---------------}(2.2)$$

$$u_{x_3 x_k =} v_{\lambda_i \lambda_t} \frac{\partial \lambda_i}{\partial x_j} \frac{\partial \lambda_i}{\partial x_k} + v_{\lambda_i} \frac{\partial^2 \lambda_i}{\partial^2 x_j dx_k}.$$

Thus, differential equation (3.1) in the new coordinates is given by

$$a^{jk}(x) \frac{\partial \lambda_i}{\partial x_j} \frac{\partial \lambda_i}{\partial x_k} + v_{\lambda_i \lambda j} + terms\, known\, on\, S_o = 0.$$

Since $u_{\lambda_k}(\lambda^1, ...\lambda_n - 1, 0), k = 1, ..., n,$ are known, see (3.2), it follows that $u_{\lambda_k \lambda j}, l = 1, ..., n-1,$ are known on $S_0 = 0.$ Thus we know all second derivatives $u_{\lambda_k \lambda t}, l = 1...., n-1,$ with the only exception of

$u_{\lambda n \lambda n}$

We recall that, provided a is sufficiently regular.

$$v_{\lambda k \lambda t}(\lambda_1, ..., \lambda_{n-1,} 0)$$

Is the limit of

$$\frac{\upsilon_{\lambda_k}(\lambda_{1,},....\lambda_l+h,\lambda_l+1,...,\lambda_{n-1,}0)-\upsilon_{\lambda_k}(\lambda_1,...,\lambda_1,\lambda_l+1,...,\lambda_{n-1},0)}{h}$$

As $h\to 0.$

Thus the differential equation can be written as

$$\sum_{j,k=1}^{n} a^{jk}(x)\frac{\partial\lambda_n}{\partial\lambda_j}\frac{\partial\lambda_n}{\partial x_k}\upsilon_{\lambda n\lambda n} = terms\,known\,on\,S_0.$$

It follows that we can calculate $\upsilon_{\lambda n\lambda n}\, if$

$$\sum_{i,j=1}^{n} a^{ij}(x)\chi_{x_i}\chi_{x_j}\neq 0\text{---------}(2.3)$$

On $S.$ This is a condition for the given equation and for the given surface $S.$

Definition: The differential equation.

$$\sum_{i,j=1}^{n} a^{ij}(x)\chi_{x_i}\chi_{x_j}=0$$

Is called characteristic differential equation associated to the given differential equation (2.1).

If $\chi,\nabla\chi\neq 0,$, is a solution of the characeritic differential equation, then the surface defined by $\chi=0$ is called characteristic surface.

<u>**Remarks:**</u> The condition (3,3) is satisfied for each χ with $\nabla\chi\neq 0$ of the quadratic matrix $(a^{ij}(x))$ is positive or negative definite for each $x\in\Omega,$ which is equivalent to the property that all eigenvalues are different from zero and have the same sign. This follows since there is a $\lambda(x)>0$ such that, in the case that the matrix (a^{ij}) is positive definite.

$$\sum_{iJ=1}^{n} a^{ij}(x)\varsigma_i\varsigma_j\geq\lambda(x)\left|\varsigma\right|^2$$

For all $\zeta\in R^n.$ .Here and in the following we assume that the matrix (a^{ij}) is real and synmetirc.

The characterization of differential equation (3.1) follows from the signs of the eign values of a $(a^{ij}(x))$

Definition. Differential equation (2.1) is said to be of type (α, β, γ) at $x \in \Omega$ if α

eigenvalues of $(a^{ij}(x))$ are positive, β eigenvalues are negative and γ eigenvalues are zero $(\alpha + \beta + \gamma = n)$.

In particular, equation is called

Elliptic if it is of type (n,0,0),i.e., all eigenvalues are different from zero and have the same sign.

Parabolic if it is of type $(n-1, 0, 1)$ or of type $(0, n-1, 1)$, i.e., on eigenvadues are different from zero and have the same sign.

Parabolic if it is of type $(n-1, 0, 1)$ or of type $(0, n-1, 1)$, i.e., one eigenvalue is zero and all the others are different from zero and have the same sign.

Hyperbolic if it is of type $(n-1, 1, 0)$ or of type $(1, n-1, 0)$, i.e., all eigenvalues are different from zero and one eigenvalue has another sign than all the others.

REMARKS:

1. According to this definition there are other types aside from elliptic, parabolic or hyperbolic equations.
2. The classification depends in general on $x \in \Omega$. An example is the Tricomi equation, which appears in the theory of transonic flows.

$$yu_{xx} + u_{yy} = 0$$

This equation as elliptic if $y > 0$, parabolic if $y = 0$ and hyperbolic for $y < 0$

Example<u>s:</u>

1. The Laplace equation in $R^3 is \nabla u = 0,$ where

$$\nabla u := u_{xx} + u_{yy} + u_{zz.}$$

Notes

This equation is elliptic. Thus for each manifold S given by $\{(x.,y,z):\chi(x,y,z)=0\}$, , where χ is an arbitrary sufficiently regular function such that $\nabla\chi\neq 0$, all derivatives of u are known on S, provided $u\, and\, \nabla u$ are known on S.

2. The wave equation $u_{xx}+u_{yy}+u_{zz}, where\, u=u(x,y,z,t),$ is hyperbolic. Such a type describes oscillations of mechanical structures, for example.
3. The heat equation $u_t=u_{xx}+u_{yy}+u_{zz}, where\, u=u(x,y,z,t),$ where $u=u(x,y,z,t),$ is parabolic. It describes, for example, the propagation of beat in a domain.
4. Consider the case that the (real)coefficients a^{ij} in equation (3,1) are constant. We recall that the matrix $A=(a^{ij})$ is symmetric, i.e., $A^T=A.$ In this case, the transform to principle axis leads to a normal form from which the classification of the equation is obviously. Let U be the associated orthogonal matrix, then

$$U^T AU=\begin{pmatrix}\lambda_1 & 0... & 0\\ 0 & \lambda_2... & 0\\ 0 & 0... & \lambda_n\end{pmatrix}$$

Here is $U=(z_1,....z_n),$,where $z_1, l=1,...,n,$, is an orthonoral system of eigenvectors to the eigenvalues λ_1.

Set $y=U^T x\, and\, \upsilon(y(=u(Uy),$ then

$$\sum_{i,j=1}^{n} a^{ij}u_{x_i x_j}=\sum_{i=1}^{n}\lambda_i\upsilon_y, y_j \qquad \text{---------(2.4)}$$

2.2.1 Normal form in two variables

Consider the differential equation

$a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy}+$ terms of lower order =0-----(2.5) in $\Omega\subset R^2$..

The associated characteristic differential equation is

$$a\chi_x^2+2b\chi_x\chi_y+c\chi_y^2=0. \qquad \text{------ (2.6)}$$

We show that an appropriate coordinate transform will simplify equation (2.5) sometimes in such a way that we can solve the transformed equation explicitly.

Let $z = \phi(x, y)$ be a solution of (2.6). Consider the level sets $\{(x, y) : \phi(x, y) = const.\}$ and assume $\phi_y \neq 0$ at a point (x_0, y_0) of the level set. Then there is a function $y(x)$ .defined in a neighborhood of x_0 such hat $\phi(x, y(x)) = const.$ it follows

$$y'(x) = -\frac{\phi_x}{\phi_y,}$$

Which implies, see the characteristic equation (2.6),

$$ay'^2 - 2by' + c = 0 \text{ --------(2.7)}$$

Then, provided $a \neq 0$, we can calculate $\mu := y'$ from the (known) coefficients a,b and c

$$\mu_{1,2} = \frac{1}{a}(b \pm \sqrt{b^2 - ac}). \text{ --------(2.8)}$$

These solutions are real if and only of $ac - b^2 \leq 0$.

Equation (3.5) is hyperbolic if $ac - b^2 < 0$. .parabolic if $ac - b^2 = 0$ and elliptic if $ac - b^2 > 0$. This follows from an easy session. Of the eigenvalues of the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Normal form of a hyperbolic equation

Let $\phi \, and \, \psi$ are solutions of the characteristic equation (2.6) such that

$$\psi_1' \equiv \mu_1 = -\frac{\phi_x}{\phi_y}$$

$$\psi_2^1 \equiv \mu_2 = -\frac{\psi_x}{\psi_y},$$

Where $\mu_1 and \mu_2$ are given by (2.8). Thus $\phi \, and \, \psi$ are solutions of the linear homogeneous equations of first order

$$\phi_x + \mu_1(x, y)\phi_y = 0 --- (2.9)$$
$$\psi_x + \mu_2(x, y)\psi_y = 0 --- (2.10)$$

Assume $\phi_x(x, y), \psi(x, y)$ are solutions such that $\nabla\phi_x \neq 0 \, and \, \nabla\psi \neq 0$, see an exercise for the existence of such solutions.

Notes

Consider two families of level sets defined by

$$\phi_x(x, y) = \alpha \; and \; \psi(x, y) = \beta$$

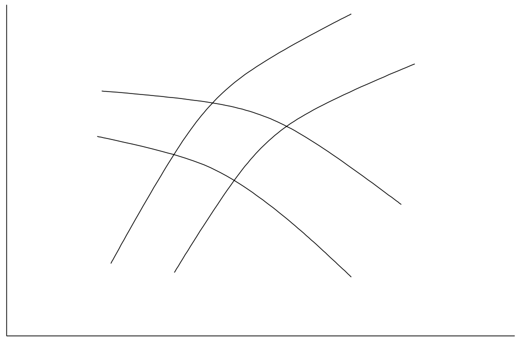


Figure 2.3 **Level sets**

These level sets are characteristic curves of the partial differential equations (3,9) and (3,10), respectively, see an exercise of the previous chapter.

Lemma. (i) Curves from different families can not touch each other.

$(ii)\, \phi_x \psi_y - \phi_y \psi_x \neq 0.$

Proof. (i):

$$y_2' - y_1' \equiv \mu_2 - \mu_1 = \frac{2}{a}\sqrt{b^2 - ac} \neq 0.$$

ii):

$$\mu_2 - \mu_1 = \frac{\phi_x}{\phi_y} - \frac{\psi_x}{\psi_y}.$$

<u>**Proposition 2.1**</u>. The mapping $\xi\mu_2 - \mu_1 = \frac{\phi_x}{\phi_y} - \frac{\psi_x}{\psi_y}$. transforms equation (2.5) into

$$\upsilon\xi_\eta =$$

Where $\upsilon(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta)).$

Proof. He proof follows from a straightforward calculation.

$$u_x = \upsilon\xi\phi_x + \upsilon_\eta\psi_x$$

$$u_y = \upsilon\xi\phi_y + \upsilon_\eta\psi_y$$

$$u_{xx} = \upsilon_{\xi\xi}\phi_x^2 + 2\upsilon_{\xi_\eta}\phi_x\psi_x + \upsilon_{\eta_\eta}\psi_x^2 + \text{lower order terms}$$

$$u_{xy} = \upsilon_{\xi\xi}\phi_x\phi_y^2 + \upsilon_{\xi_\eta}(\phi_x\psi_y + \upsilon_y\psi_x) + \upsilon_{\eta_\eta}\psi_x\psi_y + \text{lower order terms.}$$

$$u_{yy} = \upsilon_{\xi\xi}\phi_y^2 + 2\upsilon_{\xi_\eta}\phi_y\psi_y + \upsilon_{\eta_\eta}\psi_y^2 + \text{lower order terms}$$

Thus

$$au_{xx} + 2bu_{xy} + cu_{yy} = \alpha\upsilon_{\xi\eta} + \gamma\upsilon_{\eta\eta} + L.o.t.,$$

Where

$$\alpha := a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2$$

$$\beta := a\phi_x\psi_x + b(\phi_x\psi_y + \phi_y\psi_x) + c\phi_y\psi_y$$

$$\gamma := a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2.$$

The coefficients $\alpha\, and\, \gamma$ are zero since $\phi\, and\, \psi$ are solutions of the characteristic equation. Since

$$\alpha\gamma - \beta^2 = (ac - b^2)(\phi_x\psi_y - \phi_y\psi_x)^2,$$

It follows from the above lemma that the coefficient β is different from zero

<u>Example:</u> Consider the differential equation

$$u_{xx} - u_{yy} = 0$$

The associated characteristic differential equation is

$$\chi_x^2 - \chi_y^2 = 0.$$

Since $\mu_1 = -1 and \mu_2 = 1,$ the functions $\phi\, and\, \psi$ satisfy differential equations.

$$\phi_x + \phi_y = 0$$
$$\psi_x - \psi_y = 0$$

Solutions with $\nabla\phi \neq 0\, and\, \nabla\psi \neq 0$

$$\phi \neq x - y, \psi = x + y.$$

Thus the mapping

$$\xi = x - y, \eta = x + y$$

Leads so the simple equation

$$\upsilon_{\xi\eta}(\xi\eta) = 0.$$

Assume $\upsilon \in C^2$ is a solution, then $\upsilon_\xi f_1(\xi)$ for an arbitrary C^1 function $f_1(\xi).$ It follows.

$$\upsilon(\xi,\eta) = \int_0^\xi f_1(\alpha)d\alpha + g(\eta),$$

Where g is an arbitrary C^2 function. Thus each C^2- solution of the differential equation can be written as

$$(*) \quad \upsilon(\xi,\eta) = f(\xi) + g(\eta).$$

Notes

Where $f, g \in C^2$.. On the other hand, for arbitrary C^2 functions f, g the function (*) is a solution of the differential equation $u_{\xi\eta} = 0$. . Consequently each C^2 solution of the original equation $u_{xx} - u_{yy} = 0$ is given by

$u(x, y) = f(x - y) + g(x + y).$

Where $f, g \in C^2$.

Check Your Progress-1

1. Show that Maxwell equations are a hyperbolic system.

----

2. Prove formula (2.22) for the normal on a surface.

2.3 QUASILINEAR EQUATIONS OF SECOND ORDER

Here we consider the equation

$$\sum_{i,j=1}^{n} a^{ij}(x, u, \nabla u) u_{x_1 x_j} + b(x, u, \nabla u) = 0 \text{--------}(2.12)$$

In a domain $\Omega \subset R^n$, where $u : \Omega \longleftrightarrow R.$. We assume that $a^{ij} = a^{ji.}$

As in the previous section new can derive the characteristic equation

$$\sum_{ij=1}^{n} a^{ij}(x, u, \nabla u) \chi_{x_i}, \chi_{x_j} = 0$$

In contrast to linear equations, solutions of the characteristic equation depend on the solution considered.

2.3.1 Quasilinear elliptic equations

There is a large class of quasilinear equations such that the associated characteristic equation has no solution $\chi, \nabla\chi \neq 0$.

Set

$U = \{(x,z,p) : x \in \Omega, z \in R, p \in R^n\}$.

Definition : The quasilinear equation (3.12) is called elliptic if the matrix $(a^{ij}(x,z,p))$ is positive definite for each (x,z,y) $(x,z.p) \in U$.

Assume equation (3,12) is elliptic and let $\lambda(x,z,p)$ be the minimum and $\Delta(x,z,p)$ the maximum of the eigen values of (a^{ij}), then

$$0 < \lambda(x,z,p)|\zeta| \leq \sum_{ij=1}^{n} a^{ij}(x,z,p)\zeta_i\zeta_j \leq \Delta(x,z,p)|\zeta|^2$$

For all $\zeta \in R^n$.

Definition: Equation (2.12) is called uniformly elliptic if ∇ / λ is uniformly bounded in U.

As important class of elliptic equations which are not uniformly elliptic (nonuniformly elliptic) is

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\right) + lower\, order\, terms = 0. \text{ ------(2.13)}$$

The main part is the minimal surface operator (left hand side of the minimal surface equation). The coefficients a^{ij} are

$$a^{ij}(x,z,p) = (1+|p|^2)^{-1/2}\left(\partial_{ij} - \frac{p_i p_j}{1+|p|^2}\right),$$

a^{ij} denotes the Kronecker delta symbol. It follows that

$$\lambda = \frac{1}{(1+|p|^2)^{3/2}}, \Delta = \frac{1}{(1|p|^2)^{1/2}}.$$

Thus equation (2.13) is not uniformly elliptic.

The behavior of solutions of uniformly elliptic equations is similar to linear elliptic equations in contrast to the behavour of solutions of nonuniformly eflliptic equations. Typical examples for nonuniformly elliptic equations are the minimal surface equation and the capillary equation.

2.4 SYSTEMS OF FIRST ORDER

Consider the quasilinear system $\sum_{k=1}^{n} A^k(x,u)u_{u_k} + b(x,u) = 0,$ ----------

(2.14)

Where A^k are $m \times m - matrices,$ matrices, sufficiently regular with respect to their arguments, and

$$u = \begin{pmatrix} u_1 \\ . \\ . \\ . \\ u_m \end{pmatrix}, u_{x_k =} \begin{pmatrix} u_1,_{x_k} \\ . \\ . \\ . \\ u_m,_{x_k} \end{pmatrix}, b = \begin{pmatrix} b_1 \\ . \\ . \\ . \\ b_m \end{pmatrix}$$

We ask the same questions as above. Can we calculate all derivative of u in a neighbourhood of a given hypersurface $S\,in\,R^n$ define by $\chi(x) = 0, \nabla\chi \neq 0.$ provided $u(x)$ is given on S?

For an answer we map S onto a flat surface S_0 by using the mapping $\lambda = \lambda(x)$ of Section 3.1 and write equation (2.14) in new coordinates. Set $v(\lambda) = u(x(\lambda)),$ then

$$\sum_{k=1}^{n} A^k(x,u)\chi_{x_k} v_{\lambda_n.} = terms\,known\,on\,S_o$$

We can solve this system with respect $v_{\lambda_n,}$ provided that

$$\det\left(\sum_{k=1}^{n} A^k(x,u)\chi_{x_k}\right) = 0$$

Fir $\zeta \in R^n.$

<u>Definition.</u> (i) The system (2.14) is hyperbolic at $(x, u(x)).$ if there is a regular linear mapping $\zeta = Q\eta,$ where $\eta = (\eta_1, ..., \eta_n - 1, k),$ such that there exists m real roots $K_k(x, u(x), \eta_1, ...\eta_n - 1), \mathrm{k} = 1, ..., \mathrm{m}, of$

$D(xu(x), \eta_1, ..., \eta_n - 1, K) = 0$

For all $(\eta_1, ..., \eta_n - 1),$ where

$D(x, u(x), \eta_1, ..., \eta_n - 1K) = C(x, u(x), x, Q\eta).$

(ii) System (2.14) is parabolic if there exists a regular linear mapping $\zeta = Q\eta$ such that D is independent of k,I,e,. D depends on less than n parameters.

(iii) System (2.14) is elliptic if $C(x, u, \zeta) = 0\,only\,if\,\zeta = 0.$

<u>Remarks:</u> In the elliptic case all derivatives of the solution can be calculated from the given data and the given equation.

Examples:

1. Beltrami equations

$$Wu_x - bv_x - cv_y = 0 - - - (2.15)$$
$$Wu_y + au_x + bv_y = 0, - - - (2.16)$$

Where W,a,b,c are given functions depending of $(x, y), W \neq 0$ and the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is positive definite.

The beltrami system is a generalization of Cauchy-Riemann equations. The function $f(x) - u(x, y) + \mathrm{iv}(x, \mathrm{y}), \mathrm{where\, z} = x + \mathrm{iy},$ is called a quasiconform mapping, see for example $[9]$, Chapter 12, for an application to partcial diferential equations.

Set

$$A^1 = \begin{pmatrix} W & -b \\ 0 & a \end{pmatrix}, A^2 = \begin{pmatrix} 0 & -c \\ W & b \end{pmatrix}$$

Then the system (3.15), (3.16) can be written as

$$A^1 = \begin{pmatrix} u_x \\ v_x \end{pmatrix} + A^2 \begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus

$$C(x, y, \zeta) = \begin{vmatrix} W\zeta_1 & -b\zeta_1 & -c\zeta_2 \\ W\zeta_2 & -a\zeta_1 & +b\zeta_2 \\ & & \end{vmatrix} = W(a\zeta_1^2 + 2b\zeta_1\zeta_2 + c\zeta_2^2),$$

Which is different from zero if $\zeta \neq 0$ according to the above assumptions. Thus the Beltrami system is elliptic.

Maxwell equations

The Maxwell equations in the isotropic case are

$$cvot_x H = \lambda E + \in E_t \text{ ----------(2.17)}$$

$cvot_x E = -\mu H_t$, -----------(2.18)

Where

$E = (e_1, e_2, e_3)^T$ Electric field strength, $e_i = e_i(x,t), x = (x_1, x_2, x_3)$,

$H = (h_1, h_2, h_3)^T$ magnetic field strength, $h_i = h_i(x,t)$,

c speed of light,

λ specific conductivity.

$\in$ di electricity constant.

μ magnetic permeability.

Here $c, \lambda, \in and\ \mu$ are positive constants.

Set $P_0 = \chi_t, \mathrm{P}_i = \chi_{x_i}, i = 1,..,3,$ then the characteristic differential equation is

$$\begin{vmatrix} \in p0/c & 0 & & 0 & 0 & P_3 - P_2 \\ 0 & \in p0/c & 0 & -P_3 & 0 & P_1 \\ & & & & & \\ 0 & 0 & \in p_o/C & 0 & 0 & \end{vmatrix} = 0/$$

The following manipulations simplifies this equation.

(i) Multiply the first three columns with $\mu Po/C$,

(ii) Multiply the 5th column with $-P_3 and\ the\ 6th$ column with P_2 and add the sum to the 1st column.

(iii) Multiply the4th column with P_3 and the 6th column with $-P_1$ and add the sum to the 2th column.

(iv) Multiply the 4th column with P_3 and the 6th column with - P_1 and add the sum to the 2th column.

(v) Multiply the 4th column with $-P_2$ and the 5th column with $-P_1$ and add the sum to the 3th column.

(vi) expand the resulting determinant with respect to the elements of the 6th,5th and 4h row.

(vii) We obtain

$$\begin{vmatrix} q + p_1^2 & p_1 p_2 & p_1 p_3 \\ p_1 p_2 & q - p_2^2 & p_2 p_3 \\ p_1 p_3 & p_2 p_3 & Q + p_3^2 \end{vmatrix} = 0,$$

Where

$$q := \frac{\in \mu}{c^2} p_0^2 - g^2$$

With $g^2 := \mathrm{p}_1^2 + p_2^2 + p_3^2$. The evaluation of the above equation leads to

$$g^2 : (g + g^2) = 0, i.e.,$$

$$\chi_t^2 \left(\frac{\in \mu}{c^2} \chi_t^2 - |\nabla_x \chi|^2 \right) = 0.$$

It follows immediately that Maxwell equations are a hyperbolic system, see an exercise. There are two solutions of this characteristic equation. The first, one are characteristic surfaces $S(t)$, defined by

$$\chi(x,t) \equiv f(n, x - Vt) = 0,$$

here we assume that 0 is in he range of $f : R \longleftrightarrow R$. Thus, $S(t)$

is defined by n-x-Vt-c, where c is a fixed constant. In follows that the planes $S(t)$ is defined by $nx - Vt - c,$ where c is a fixed constant. It follows that the planes $S(t)$ with normal n move with speed V in direction of n, are Figure 2.4

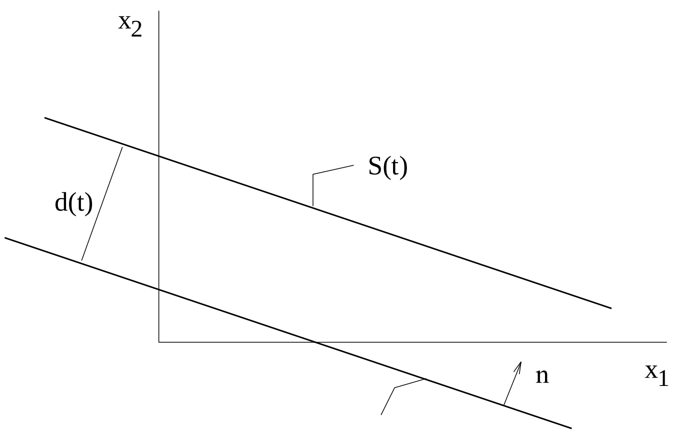


Figure 2.4: $d^l(t) is\, the\, speed\, of\;\, plane\, awaves$

<u>Remark:</u> According to the previous discussions, singularities of a solution of Maxwell equations are located at most on characteristic surface.

A special case of Maxwell equations are the Telegraph equations. Which follow from Maxwell equation if div E=0 and div H=0 and div H=0,i.e., E and H are fields free of sources. In fact, it is sufficient to assume that this assumption is satisfied at a fixed time to only, see an exercise.

Since

$$rot_x rot_x A = grad_x\, div_x A - \nabla_x A$$

For each C^2 – vector field A, it follows from Maxwell equations the uncoupled system

$$\Delta_x E = \frac{\in \mu}{c^2} E_{tt} + \frac{\lambda\mu}{c^2} E_t$$

$$\Delta_x H = \frac{\in \mu}{c^2} H_{tt} + \frac{\lambda\mu}{c^2} H_t$$

Equations of gas dynamics

Consider the following quasilinear equations of first order

$$v_1 + (v.\nabla_x)v + \frac{1}{p}\nabla_x p = f \text{ (Euler equations).}$$

Here is

$v = (v_1 v_2, v_3)$ the vector of speed, $v_1 = v_2(x,t), x = (x_1, \mathrm{x}_2, \mathrm{x}_3),$

pressure, $p = (x,t),$

p $density, p - p(x,t),$

$f = (f_1, f_2 f_3)$ density of the external force, $f_i = f_i(x,t),$

$(v.\nabla_x)v \equiv (v.\nabla_x v_1, v.\nabla_x v_2, v, \nabla_x v_3))^T.$

The second equation is

$pt + v.\nabla_x p + pdiv_x v = 0$ (conservation of mass).

Assume the gas is compressible and that ther is a function (state equation)

$p = p(p),$

Where $P^{'}(P) > 0 \,\text{if p} > 0.$ Then the above system of four equations is

$$v_t + (v.\nabla)\,\mathrm{v} + \frac{1}{p}\,\nabla^{'}(p)\nabla p = f ---(2.19)$$

$$p_1 + p\ divv + v\ \nabla p = 0. ---(2.20)$$

Where $\nabla \equiv \nabla_x and\ div \equiv div_{x,}$ i.e., these operators apply on the spatial variables only.

The characteristic differential equation is here

$$\begin{vmatrix} \frac{dx}{dt} & 0 & 0 & \frac{1}{\rho}\rho^1 x_1 \\ 0 & \frac{dx}{dt} & 0 & \frac{1}{\rho}\rho^1 x_2 \\ 0 & 0 & \frac{dx}{dt} & \frac{1}{\rho}\rho^1 x_3 \\ \rho X x_1 & \rho X x_2 & \rho X x 3 & \frac{dx}{dt} \end{vmatrix} = 0$$

Where

$$\frac{dx}{dt} = X_t + (\nabla_x X).V$$

Evaluating the determinant, we get the characteristic differential equation

$$\left(\frac{d\chi}{dt}\right)^2 \left(\left(\frac{d\chi}{dt}\right)^2 - p^{'}(p)\left|\nabla_x \chi\right|^2\right) = 0. \text{-----------}(2.21)$$

The equation implies consequences for the speed of the characteristic surfaces as the following consideration shows.

Consider a family $S(t)$ of surfaces in R^3 defined by $\chi(x,t) = \mathrm{c}$, where $x \in R^3\, and\, c\, is\, a\ fixed$ constant. As usually, we assume that $\nabla_x \chi \neq 0,$ One of the two normals on $S(t)$ at a point of the surface $S(t)$ is given by, see an exercise.

$$n = \frac{\nabla_x \chi}{\left|\nabla_x \chi\right|}. \text{---------}(2.22)$$

Let $Q_0 \in S(t_0) and\ let\, Q_1 \in S(t_1)$ be a point on the line defined by $Q_0 + sn,$ where n is the normal (3.22 on $S(t_0) at\, Q_o and\, t_0 < t_1, t_1 - t_0$ small, we

Notes

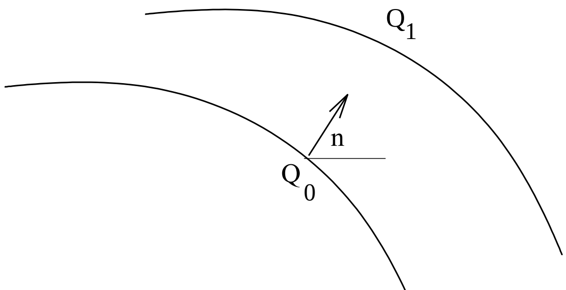


Figure 2.5 Definition of speed of a surface

Definition :. The limit

$$P = \lim_{t_1 \to d_0} \frac{|Q_1 - Q_0|}{t_1 - t_0}$$

Is called speed of the surface $S(t)$.

Proposition 2.3: The speed of the surface $S(t)$ is

$$P = -\frac{\chi t}{|\nabla_x \chi|} ...(2.23)$$

Proof. The proof follows from $\chi(Q_0, t_0) = 0 \, and \, \chi(Q_0 + dn, t_0 + \Delta t) = 0.$

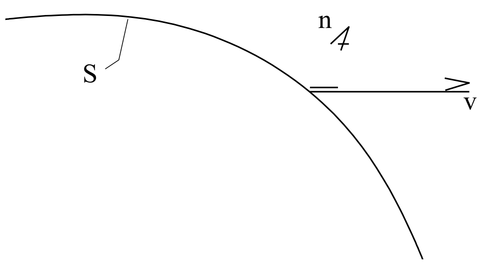
where $d = |Q_1 - Q_0| \, and \, \Delta t = t_1 - t_0$

Set $v_n := v.n$ In which is the component of the velocity vector in directions n. From (2.22) we get

$$v_n \frac{1}{|\nabla_x \chi|} v.\nabla_x \chi$$

Definition: $V := P - v_n,$ the difference of the speed of the surface and the speed of liquid particles, is called relative speed

Figure 2.6: Definition of relative speed



Using the above formulas for P and u_n it follows

$$V = P - v_n = -\frac{\chi t}{|\nabla_x \chi|} - \frac{v\nabla_x \chi}{|\nabla_x \chi|} = -\frac{1}{\nabla_x \chi}\frac{d\chi}{dt}$$

Them, we obtain from the characteristic equation (3.21) that

$$V^2 = |\nabla_x \chi|^2 (V^2 |\nabla_x \chi|^2 - p'(p)|\nabla_x \chi|^2) = 0$$

An interesting conclusion is that there are two relative speeds. V=0

$V^2 = P'(P).$

Definition : $\sqrt{p'(p)}$ is called speed of sound.

Check your progress

3. Discuss about linear equations of second order.

2.5 SYSTEMS OF SECOND ORDER

Here we consider the system

$$\sum_{k,J=1}^{n} A^{kl}(x,u,\nabla u)^{u_x k} n_1 + \text{lower order terms} = 0,\text{-------}(2.24)$$

Where, $A^{kt} are (m \times m)$ matrices and $u = (u_1, ...u_m)^T$. We assume $A^{kl} = A^{lk}$, . Which is no restriction of generality provided $u \in C^2$ is satisfied. As in the previous sections , the classification follows from the question where or not we can calculate formally the solution from the differential equations, if sufficiently many data are given on an initial manifold. Let the initial manifold. Let the initial manifold S be given by $\chi(x) = 0$ and assume that $\nabla \chi \neq 0.$. The mapping $x = x(\lambda).$, see previous sections, leads to

$$\sum_{kJ=1}^{n} A^{kj} \chi_{xk} \chi_{xt} p_{\lambda_0, \lambda_0,} = terms\, known\, on\, S.$$

Notes

Where $v(\lambda) = u(x(\lambda))$.

The charactreristic equation is here

$$\det\left(\sum_{k,j=1}^{n} A^{kt}\chi_{x_k}\chi_{xt}\right) = 0.$$

If there is a solution χ with $\nabla\chi \neq 0$, then it is possible that second derivatives are not continuous in a neighbor hold of S .

Definition : The system is called elliptic if

$$\det\left(\sum_{k,j=1}^{n} A^{kt}\zeta_{x_k}\zeta_{xt}\right) \neq 0$$

For all $\zeta \in \square^{o}, \zeta \neq 0.$

2.6.1 Examples :

Navier –Stokes equations:

The navier-Stokes system for a viscous incompressible liquid is

$$v_t + (v.\nabla_x)v = -\frac{1}{0}\nabla_x p + \gamma\Delta_x v$$

$$div_x v = 0.$$

Where p is the (constant and positive) density of liquid

γ is the (constant and positive) viscosity of liquid.

$v = v(x,t)$ velocity vector of liquid particles, $x \in \square^3 or\, in \square^3$,

$p = p(x,t)$ pressure

The problem is to find solutions u,p of the above system.

Linear elasticity:

Consider the system

$$p\frac{\partial^2 u}{\partial t^2} = \mu\Delta_x u + (\lambda+\mu)\nabla_x(div_x u) + f. \text{-------}(2.25)$$

Here is, in the case of an elastic body in R^3,

$u(x,t) = (\mathrm{v}_1(x,t), v_2(x,t), v_3(x,t))$ displacement vector,

$f(x,t)$ density of external force,

p (constant) density

λ (positive) Lame constants.

The characteristic equation is det C-9, wehre the entries of the matrix C are given by

$$c_{xj} = (\lambda+\mu)\chi_{x,}\chi_{j,} + \partial_{ij}(\mu|\nabla_x\chi|^2 - p\chi_1^2).$$

The characteristic equation is

$$((\lambda+2\mu)|\nabla_x\chi|^2 - p\chi_t^2(\mu|\nabla_x\chi|^2 - p\chi_t^2).$$

It follows that two different speeds P of characteristic surfaces $S(t)$, defined by $\lambda(x,t)=const.$., are possible, namely

$$P_1=\sqrt{\frac{\lambda+2\mu}{p}},\,and\;P_2=\sqrt{\frac{\mu}{p}}.$$

We recall that $P=-\chi_t/|\nabla_x\chi|$.

Exercise:

1. Show that the differential equation

$a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy}$ +lower term=0

Is elliptic if $ac-b^2>0$, parabolic if $ac-b^2=0$ and hyperbolic if $ac-b^2<0$

2. Show that hyperbolic case there exits a solution of

$\phi_x+\mu_1\phi_y=0$, see equation (2.9)

Such that $\nabla\phi\neq 0$.

Hint: Consider an appropriate Cauchy initial value problem.

3. Determine the type of the following equation at (x, y) = (1, 1/2).

$Lu := xu_{xx} + 2yu_{xy} + 2xyu_{yy} = 0$.

4. Find all C^2 solutions of $u_{xx}-4u_{xy}+u_{yy}=0$

Hint: Transform to principal axis and stretching of axis lead to the wave equation.

5. Determine the type of $u_{xx} - xu_{yx} + u_{yy} + 3u_x = 2x$, where u = u(x, y).

6. Transform equation $u_{xx}+(1-y^2)u_{xy}=0$, u = u(x, y), into its normal form.

7. Transform the Tricomi-equation $yu_{xx} + u_{yy} = 0$, u = u(x, y), where y < 0, into its normal form.

2.6 LET US SUM UP

Notes

In this chapter we have discussed about The general non linear partial differential equation of second order. Elliptic if it is of type (n,0,0), Parabolic if it is of type, Parabolic if it is of type $(n-1,0,1)$ or of type $(0,n-1,1)$, Hyperbolic if it is of type $(n-1,1,0)$ or of type $(1,n-1,0)$. There is a large class of quasilinear equations such that the associated characteristic equation has no solution. In the elliptic case all derivatives of the solution can be calculated from the given data and the given equation. Cauchy-Riemann equations Cauchy-Riemann equations. Maxwell equations. The Navier-Stokes equations.

2.7 KEY WORDS

1. The general nonlinear partial differential equation of second order is $F(x,u,Du,D^2v)=0.$
2. Normal form in two variables
3. Of the eigenvalues of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$
4. Curves from different families can not touch each other.
5. Solutions of the characteristic equation depend on the solution considered.
6. There is a large class of quasilinear equations such that the associated characteristic equation has no solution $\chi, \nabla\chi \neq 0.$
7. Beltrami equations
8. Maxwell equations

2.8 QUESTIONS FOR REVIEW

1. Show that Maxwell equations are a hyperbolic system.
2. Consider Maxwell equations and prove that div E = 0 and div H = 0 for all t if these equations are satisfied for a fixed time t_0.
   Hint. div rot A = 0 for each C^2-vector field $A=(A_1, A_2, A_3)$
3. Prove formula (2.22) for the normal on a surface.
4. Prove formula (3.23) for the speed of the surface S(t).
5. Discuss about linear equations of second order.

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2.10 ANSWERS TO CHECK YOUR PROGRESS

1. See section 2.2
2. See section 2.2
3. See Section 2.5

UNIT – 3 WAVES AND DIFFUSIONS

STRUTURE

3.0 OBJECTIVES

After studying this unit you should be able to learn and understand about

The wave equation, Causality and energy, The diffusion equation, Diffusion on the whole line,

Comparison of waves and diffusion.

3.1 INTRODUCTION

In this chapter we study the wave and diffusion equations on the whole real line $-\infty < x < +\infty$. Real physical situations are usually on finite intervals. We are justified in taking x on the whole real line for two reasons. Physically speaking, if you are sitting far away from the boundary, it will take a certain the solutions we obtain in this chapter are valid. Mathematically speaking, the absence of a boundary is a big simplification. The most fundamental properties of the PDEs can be found most easily without the complications of boundary conditions. That is the purpose of this chapter. We begin with the wave equation.

3.2 THE WAVE EQUATION

We write the wave equation as

$$\boxed{u_{tt} = c^2 u_{xx} \ for - \infty < x < +\infty} \qquad (1)$$

(Physically, you can imagine a very long string.) This is the simplest second-order equation. The reason is that the operator factors nicely:

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0.$$

(2)

This means that, starting from a function of (x,t), you compute $u_t + cu_x$, call the result v, then you compute $u_t + cu_x$, and you ought to get the zero function. The general solution is.

$$\boxed{u(x,t) = f(x+ct(+g(x-ct)}$$

(3)

Where f and g are two arbitrary (twice differentiable) functions of a single variable.

Proof. Because of (2), if we let $v = u_1 + cu_x$, we must have $v_1 - cu_x = 0.$ Thus we have two first-order equations.

$$v_1 - cu_x = 0 \qquad (4a)$$

And

$$u_t + cu_x = v. \qquad (4b)$$

These two first-order equations are equivalent to (1) itself. Let's solve them one at a time. As we know from Section 1,2, equation (4a) has the solution $v(x,t) = h(x+ct),$,where h is any function.

So we must solve the other equation, which now takes the form

Notes

$$u_t + cu_x = h(x+ct)$$

(4c)

for the unknown function $u(x,t)$. It is easy to check directly by differentiation that one solution is (‘) denotes the derivative $u(x,t) = f(x+ct)$, where $f(x) = h(s)/2c$. [A prime derivative of a function of one variable.] To the solution $f(x+ct)$ we can add $g(x-ct)$ to get another solution (since the equation is linear). The most general solution of (4b) in fact turns out to be a particular solution plus any solution of the homogeneous equation; that is.

$$u(x,t) = f(x+ct) + g(x-ct),$$

As asserted by the theorem. The complete justification is left to be worked out in Exercise-4.

A different method to derive the solution formula (3) is to introduce the characteristic coordinates

$$\xi = x+ct \qquad \eta = x-ct/$$

By the chain in rule, we have $\partial_x = \partial_\xi + \partial_\eta \; and \; \partial_t = c\partial_\xi + c\partial_\eta$. Therefore, $\partial_t - c\partial_x = -2c\partial_\eta \; and \; \partial_t + c\partial_x = 2c\partial_g$. So equation (1) takes the form

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (-2c\partial_g)(2c\partial_\eta)u = 0.$$

Which means that $ug_\eta = 0 \sin ce c \neq 0$. The solution of this transformed equation is $u = f(\xi) + g(\eta)$ The wave equation has a nice simple geometry. There two families of characteristic lines, $x \pm ct =$ constant, as indicated in figure 1. The most general solution is the sum of two function. One, $g(x-ct)$, is a wave of arbitrary shape traveling to the right at speed c. The other, $f(x+ct)$, is a wave of arbitrary shape traveling to the right at speed c. The other, $f(x+ct)$, is another shape traveling to the left at speed c. A “movie” of $g(x-ct)$ is sketched in Figure 1 of Section 1,3.

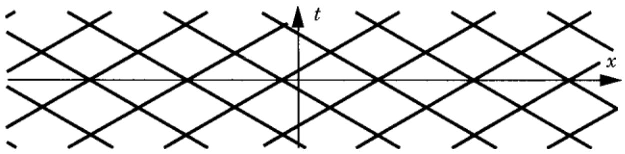


Figure (1)

INITIAL VALUE PROBLEM

The initial –value problem is to solve the wave equation

$$\boxed{u_u = c^2 u_{xx} \quad for - \infty < x, + \infty} \qquad (1)$$

With the initial conditions

$$\boxed{u(x,0) = \phi(x) \qquad u_1(x,0) = \psi(x),}$$

(5)

Where $\phi(x) = \mathrm{f(x)} + \mathrm{g(x)}.$ (6)

Then, using the chain rule, we differentiate (3) with respect to t and put t=0 to get

$$\psi(x) = cf^{'}(x) - cg^{'}(x). \qquad (7)$$

Let's regard (6) and (7) as two equations for the two unknown functions f and g. To solve them, it is convenient temporarily to change the name of the variable to some neutral name; we change the name of x to s. Now we differentiate (6) and divide (7) by c to get

$$\phi^{'} = f^{'} + g^{'} \quad and \frac{1}{c}\psi = f^{'} - g^{'}.$$

Adding and subtracting the last pair of equations gives us

$$f^{'} = \frac{1}{2}\left(\phi^{'} + \frac{\psi}{c}\right) and\ g^{1} = \frac{1}{2}\left(\phi^{'} - \frac{\psi}{c}\right).$$

Integrating, we get $f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^x \psi + A$

And $g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^x \psi + B.$

Notes

Where A and B are constants. Because of (6), we have A+B=0. This tells us what f and g are in the general formula (3). Substituting s=x+ct into the formula for f and s=xct into that of g, we get

$$u(x,t)=\frac{1}{2}\phi(x+ct)+\frac{1}{2c}\int_0^{x+ct}\psi+\frac{1}{2}\phi(\mathrm{x-ct})-\frac{1}{2c}\int_0^{x-ct}\psi.$$

This simplifies to

$$\boxed{u(x,t)=\frac{1}{2}[\phi(x+ct)+\phi(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds} \qquad (8)$$

This is the solution formula for the initial-value problem, due to d' Alembert in 1746.

Assuming ϕ to have a continuous second derivative $(written\,\phi\in C^2)$ and ψ to have a continuous first derivative $(\psi\in C^1)$, we see from (8) that u itself has continuous second partial derivatives in x and t $(\psi\in C^2)$. Then (8) is a bona fide solution of (1) and (5). You may check this directly by differentiation and by setting t=0.

Example 1.

For $\phi(x)=0\,and\,\psi(x)=\cos x$, the solution is u(x,t)=(1/2c) $[\sin(x+ct)-\sin(x-ct)]=(1/c)\cos x\sin ct$. Checking this result directly, we have $u_n=-c\cos x\sin c\,tu_{xx}=-(1/c)\cos \mathrm{x}\sin \mathrm{ct}, so\,that\,u_{tt}=v^2u_{xx}$. The initial condition is easily checked.

Example 2. The Plucked String

For a vibrating string the speed is $c=\sqrt{T/\mathrm{p}}$. Consider an infinitely long string with initial position

$$\phi x=\begin{cases} b-\dfrac{b|x|}{a} & for\,|x|<a \\ & for\,|x|>a \\ 0 & \end{cases}$$

(9)

And initial velocity $\psi(x) = 0$ for all x. This is a "three-finger" pluck, with all three fingers removed at once. A "movie" of this solution $\frac{1}{2}[\phi(x+ct) + \phi(x-ct)]$ is shown in Figure 2 (Even though his solution is not twice differentiable, it can be shown to be a "weak" solution, as discussed later.)

Each of these pictures is the sum of two triangle functions, one moving to the right and one to the left ,as is clear graphically. To write down the formulas that correspond to the pictures requires a lot more work. The formulas depend on the relationships among the five numbers

$x < -3a/2, then\, x \pm ct = x \pm a/2. First, if\ x < 3a/2, then\, x \pm a2 >= a\, amdi)x.t\_ = 0.Secon$

$$u(x,t) = \frac{1}{2}\phi\left(x + \frac{1}{2}a\right) = \frac{1}{2}\left(b - \frac{b\left|x + \frac{1}{2}a\right|}{a}\right) = \frac{3b}{4} + \frac{bx}{2a}$$

Third, if $|x| < a/2, then$

$$u(x,t) = \frac{1}{2}\left[\phi\left(x + \frac{1}{2}a\right) + \phi\left(x - \frac{1}{2}a\right)\right]$$

$$= \frac{1}{2}\left[b - \frac{b(x + \frac{1}{2}a)}{a} + b - \frac{b(\frac{1}{2}a - x)}{a}\right]$$

$$= \frac{1}{2}b$$

Notes

And so on [see Figure 2].

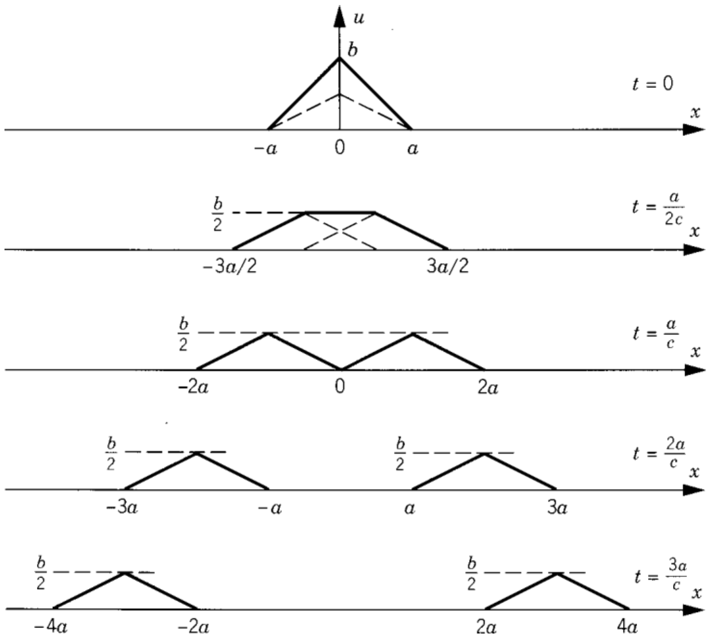


Figure (2)

EXERCISE

1. Solve $u_{tt} = c^2 u_{xx}$, $u(x, 0) = e^x$, $u_t (x, 0) = \sin x$.

2. Solve $u_{tt} = c^2 u_{xx}$, $u(x, 0) = \log(1 + x^2)$, $u_t (x, 0) = 4 + x$.

3. The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $l/4$ from one end. (Assume that $a < l/4$; otherwise, poor flea!) How long does it take for the disturbance to reach the flea?

4. Justify the conclusion at the beginning of Section 3.1 that *every* solution of the wave equation has the form $f(x + ct) + g(x - ct)$.

5. If both φ and ψ are odd functions of x, show that the solution $u(x, t)$ of the wave equation is also odd in x for all t.

6. Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t)$.

3.3 CAUSALITY AND ENERGY

CAUSALITY

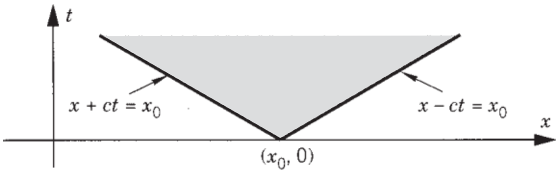


Figure (1)

We have just learned that the effect of an initial position $\phi(x)$ is a pair of waves traveling in either direction at speed c and at half the original amplitude.

The effect of an initial velocity ψ is a wave spreading out at speed $\leq c$ in both directions. So part of the wave may lag behind (if there is an initial velocity), but no part goes faster than speed c. The last assertion is called the principle of causality.

It can be visualized in the xt plane in Figure 1.

An initial condition (position or velocity or both) at the point $(x_0, 0)$ can affect the solution for t>0 only in the shaded sector, which is called the domain of influence of the point $(x_0, 0)$.

As a consequence, if $\phi \, and \psi$ vanish for $|x| > R, then\, u(x,t) = 0 \; for |x| > R + ct$. In words, the domain of influence of an interval $(|x| \leq R) is\, a \sec tor (|x| \leq R + ct)$.

An "inverse" way to express causality is the following. Fix a point (x,t) for t>0 (see figure 2). How is the number u(x,t) synthesized from the initial data ϕ, ψ? It depends only on the values of

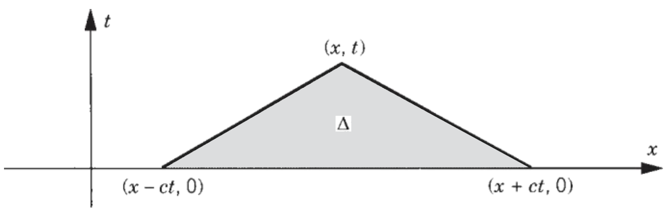


Figure (2)

Notes

ψ at the two points $x \pm ct$, and it depends only on the values of ψ within the interval [x-ct,x+ct, and it depends only on the values of ψ within the interval $[x-ct, x+ct]$. We therefore say that the interval $(x-ct, x+ct)$ is the interval of dependence of the point (x,t)on t=0. Sometimes we call the entire shaded triangle Δ the domain of dependence or the past history of the point (x,t). The domain of dependence is bounded by the pair of characteristic lines that pass through (x,t).

ENERGY

Imagine an infinite string with constants $p\, and\, T$. Then $pu_{tt} = Tu_{xx}$ for $-\infty < x < +\infty$ From physics we know that the kinetic energy is $\frac{1}{2}mv^2$, which in our case takes the form $KE \frac{1}{2} p \int u_t^2 dx$. This integral, and the following ones, are evaluated from $-\infty < x < +\infty$. To be sure that the integral converges, we assume that $\phi(x) and \psi(x)$ vanish outside an interval $\{|x| \le R\}$. As mentioned above, u(x,t) [and therefore $u_1(x,t)$] vanish for $|x| > R + ct$. Differentiating the kinetic energy, we can pass the derivative under the integral sign (see Section A,3) to get

$$\frac{dKE}{dt} = p \int u_t u_{tt} dx.$$

Then we substitute the PDE $pu_{tt} = Tu_{xx}$ and integrate by parts to get

$$\frac{dKE}{dt} = T = Tu_t u_x - T \int u_{tt} u_x dx.$$

The term $Tu_t u_x$ is evaluated at $x = \pm\infty$ and so it vanishes. But the final term is a pure derivative since $u_{tx} u_x = (\frac{1}{2} u_x^2)_t$. Therefore.

$$\frac{dKE}{dt} = -\frac{d}{dt} \int \frac{1}{2} T u_x^2 dx.$$

Let $\mathrm{PE} = \frac{1}{2} T \int u_x^2 dx\, and\, let\, E = KE + PE$. Then

$dK\,\mathrm{E/\,dt = -dPE/\,dt, or dE/\,dt = 0}$. Thus

$$\boxed{E = \frac{1}{2}\int_{-\infty}^{+\infty}(pu_t^2 + Tu_x^2)dx} \qquad (1)$$

Is a constant independent of t. This is the law of conservation of energy.

In physics courses we learn that PE has the interpretation of the potential energy. The only thing we need mathematically is the total energy E. The conservation of energy is one of the most basic facts about the wave equation. Sometimes he definition of E is modified by a constant factor, but that does not affect its conservation. Notice that the energy is necessarily positive. The energy can also be used to derive causality .

Example -1.

The plucked string, Example 2 of Section 2.1. has the energy

$$E\frac{1}{2}T\int \phi_x^2 dx = \frac{1}{2}T\left(\frac{b}{a}\right)^2 2a = \frac{Tb^2}{a}$$

In electromagnetic theory the equations are Maxwell's. Each component of the electric and magnetic fields satisfies the (three-dimensional) wave equation, where is the speed of light. The principle of causality, discussed above, is the cornerstone of the theory of relativity. It means that a signal located at the position x_0 at the instant t_0 cannot move faster than the speed of light The a signal of speed c starting from the point x_0 at the t_0 . It turns out that actually equal to c and never slower. Therefore, the causality principle is sharper in three dimension than in one. This sharp form is called Huygens's principle.

Flatland is an imaginary two –dimensional world. You can think of yourself as a water bug confined to the surface of a pond. You would n't want to live here because Huygens' 's principle is not valid is two dimensions . Each sound you make would automatically mix with the "echoes" of your previous sounds. And each view would be mixed fully with the previous views. Three is the best of all possible dimensions.

Exercise:

1. Use the energy conservation of the wave equation to prove that the only solution with $\varphi= 0$ and $\psi = 0$ is $u= 0$. (*Hint:* Use the first vanishing theorem)

2. Show that the wave equation has the following invariance properties.

a. Any translate $u(x-y, t)$, where y is fixed, is also a solution.

b. Any derivative, say u_x, of a solution is also a solution.

c. The dilated function $u(ax, at)$ is also a solution, for any constant a.

3. If $u(x, t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity

$$u(x + h, t + k) + u(x - h, t - k) = u(x + k, t + h) + u(x - k, t - h)$$

for all x, t, h, and k. Sketch the quadrilateral Q whose vertices are the arguments in the identity.

3.4 THE DIFFUSION EQUATION

In this section we begin a study of the one-dimensional diffusion equation.

$$u_t = ku_{xx}. \qquad (1)$$

Diffusions are very different from waves, and this is reflected in the mathematical properties of the equations. Because (1) is harder to solve that the wave equation, we begin this section with a general discussion of some of the properties of diffusions. We begin with the maximum principle, from which we'll deduce the uniqueness of an initial –boundary problem.

We postpone until the next section the derivation of the solution formula for(1 on the whole real line.

Maximum Principle: If $u(x,t)$ satisfies the diffusion equation in a rectangle $(say, o \le x \le l, 0 \le t \le T)$ in space-time, then the maximum value of $u(x,t)$ is assumed either initially (t=0) or on the lateral sides $(x = 0 \ or \ x = l)$ (see Figure1).

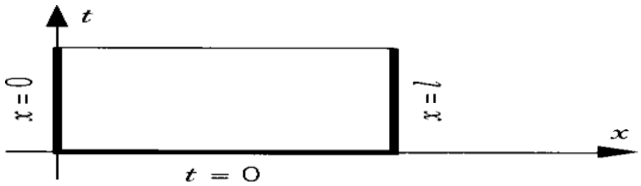


Figure (1)

In fact, there is a stronger version of the maximum principle which asserts that the maximum cannot be assumed anywhere inside the rectangle but only on the bottom or the lateral sides (unless u is a constant). The corners are allowed.

The minimum value has the same property; it too can be attained only on the bottom or the lateral sides. To prove the minimum principle. Just only the maximum principle to $[-u(x,t)].[-u(x,t)]$.

These principles have a natural interpretation in terms of diffusion or heat flow. If you have a rod with no internal heat source, the hottest spot and the coldest spot can occur only initially or at one of the two ends of the rod. Thus a hot spot at time zero will cool off (unless heat is fed into the rod at an end. You can burn one of its ends but the maximum temperature will always be at the hot end, so that it will be cooler away from that end. Similarly, if you have a substance diffusing along a tube, its higher concentration can occur only initially or at one of the ends of the tube.

If we draw a "movie" of the solution, the maximum drops down while the minimum comes up. So the differential equation tends to smooth the solution out. (This is very different from the behavior of the wave equation').

Proof of the Maximum Principal: We'll prove only the weaker version. (Surprisingly, its strong form is much more difficult to prove.) For the strong version. See [PW]. The idea of the proof is to use the fact, from calculus, that at an interior maximum the first derivatives vanish and the second derivatives vanish and second derivatives satisfy inequalities such as $u_{tt} \leq 0.$. If we knew that $u_{tt} \neq 0$ at the maximum (Which we do not), then we'd have $u_{tt} < 0$ as well as so that $u_t \neq ku_{xx}$. This contradiction would show that the maximum could only be somewhere on the boundary of the rectangle. However, because could in fact be equal to zero,

we need to play a mathematical game to make the argument work.

So let M denote the maximum value of u(x,t) on the three sides t = 0, x = 0, and x = l. (Recall that any continuous function on any bounded closed set is bounded and assumes its maximum on that set). We must show that $u(x,t) \leq M$ throughout the rectangle R.

Let $\in$ be a positive constant and let $v(x,t) = \mathrm{u(x,t)} + \mathrm{cx}^2$. Our goal is to show that $v(x,t) \leq M + cl^2$ throughout R. Once this is accomplished, we'll have $v(x,t) \leq M + \in (l^2 - x^2)$. This conclusion is true for any $\in > 0$. Therefore, $u(x,t) \leq M$ throughout R, which is what we are trying to prove. Now from the definition of v ,it is clear that $v(x,t) \leq M + \in l^2 on\, t = 0.$ on x = 0, and on x = l. this function v satisfies

$$v_1 - kv_{xx} = u_1 - k(u + \in x^2)_{xx} = -u_1 - ku_{xx} - 2 \in k = -2 \in k < 0,$$

What is the "diffusion inequality." Now suppose that v(x,t) attains its maximum at an interior point $(x_0 t_0)$. That is, $0 < x_0 < l, 0 < t_0 < T$. By ordinary calculus, we know that $v_1 = 0\, and\, v_{xx} \leq 0\, at\, (x_0, t_o)$ .This contradicts the diffusion inequality (2). So there can't be an interior maximum. Suppose now that v(x,t) has a maximum (in the closed rectangle) at a point on the top edge $\{t_0 = T\, and\, 0 < x < l\}$. Then $v_x(x_0, t_0) = 0 and\, v_{xx}(x_0, t_0) \leq 0,$ as before. Further more, because $v_x(x_o, t_0)$ is bigger than $v(x_0, t_0 - \partial),$ we have

$$v_t(x_o, t_0) = \lim \frac{v(x_0, t_0) - v(x_0, t_0 - \partial)}{\partial} \geq 0$$

As $\partial \to 0$ through positive values. (This is not an equality because the maximum is only "one sided" in the variable t.) We again reach a contradiction to the diffusion inequality.

But v (x,t) does have a maximum somewhere in the closed rectangle $0 \le x \le l, 0 \le t \le T$. This maximum must be on the bottom or sides.

Therefore $v(x,t) \le \mathrm{M} + \in \mathrm{l}^2$ throughout R. this proves the maximum principle (in its weaker version).

UNIQUENESS

The maximum principle can be used to give a proof of uniqueness for the Dirichlet problem for the diffusion equation. That is, there is at most one solution of

$$\begin{aligned} &u_t - ku_{xx} = f(x,t) \, for 0 < x < 1 and t > 0 \\ &u(x,0) = \phi(x) \\ &u(0,t) - g(t) \qquad u(l,t) = h(t) \end{aligned}$$

For four given functions $f, \phi, g, and\ h.$ Uniqueness means that any solution is determined completely by it initial and boundary conditions. Indeed, let $u_t(x,t) \, and \, u_2(x,t)$ be two solutions of (3). Let $w = u_t - u_2$ be their difference. Then

$w_1 - kw_{xx} = 0, w(x,0) = 0, w(o,t) = 0, w(l,t) = 0. Let\, T > 0.$

By the maximum principle, w(x,t) has its maximum for the rectangle on its bottom or sides – exactly where it vanishes. So w(x,t) $w(x,t) \le 0.$ Therefore, $w(x,t) = 0.$ so that $u_t(\mathrm{x,t}) = \mathrm{u}_2(x,t) \, for\, all\, t \ge 0.$

Here is a proof of uniqueness for problem (3), by a very different technique, the energy method. Multiplying the equation for $w = u_1 - u_2$ by w itself, we can write

$$0 = 0, w = w_t - kw_{xx})(w) = (\frac{1}{2} w^2)_t + (-kw_x w)_x + kw^2{}_x.$$

(verify this by carrying out the derivatives on the right side .) upon integrating over the interval $0 < x < l,$ we get

$$0 = \int_0^l \left(\frac{1}{2} w^2 \right)_t dx - kw_x w \Big|_{x=0}^{x=1} + K \int_0^1 w_x^2 dx.$$

Because of the boundary conditions (w = 0 at x = 0,l

$$\frac{d}{dt}\int_0^1 \frac{1}{2}[w(x,t)]^2 dx = -k\int_0^1 [w_x(x,t)]^2 dx \le 0,$$

Where the time derivative has been pulled out of the x integral (see section a.3). Therefore, is decreasing, so

$$\int_0^1 [w(x,t)]^2 dx \le \int_0^1 [w(x,0)]^2 dx$$

For $t \ge 0.$ The right side of (4)vanishes because the initial conditions of u and v are the same, so that $\int [w(x,t)^2]dx \le \int_0^1 [w(x,0)]^2 dx$ for all t >0. So and for all w=0 and $u_1 \equiv u_2 \; for\, all\, t \ge 0.$

STABILITY

This is the third ingredient of well-posedness .It means that the initial and boundary conditions are correctly formulated. The energy method leads to the following form of stability of problem(3), in case $h = g = f = 0. Let\, u_1(x,0) = \phi_1(x)\, and\, u_2(x,0) = \phi_2(x).$

Then $w = u_1 - u_2$ is the solution with the initial datum $\phi_1 = \phi_2(x).$ So from (4) we have

$$\int_0^1 [u_1(x,t) - u_2(x,t)]^2 dx \le \int_0^1 [\phi_1(x) - \phi_2(x)]^2 dx.$$

On the right side is a quantity that measures the nearness of the initial data for two solutions, and on the left we measure the nearness of the solutions at any later time. Thus, if we start nearby (at t = 0), we stay nearby. This is exactly the meaning of stability in the “square integral” sense. The maximum principle also proves the stability, but with a different way to measure nearness. Consider two solutions of (3) in a rectangle. We then have $w \equiv u_1 - u_2 = 0$ on the bottom. The maximum principle asserts that throughout the rectangle

$$u_1(x,t) - u_2(x,t) \le \max|\phi_1 - \phi_2|.$$

The” minimum “ principle says that

$$u_1(x,t) - u_2(x,t) \ge \max|\phi_1 - \phi_2|.$$

Therefore,

Max $\quad |u_1(x,t)-u_2(x,t) \le \max|\phi_1(x)-\phi_2(x)|.$

Valid for all t >0. Equation (6) is in the same spirit as (5), but with a quite different method of measuring the nearness of functions. It is called stability in "uniform stage" sense.

Exercise:

1. Consider the solution $1-x^2-2kt$ of the diffusion equation. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \le x \le 1, 0 \le t \le T\}$

2. Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1, 0 < t < \infty\}$ with u=$(0,t)=u(1,t)$ =0 and u(x,0)=4x(1-x).

Show that 0<u(x, t)<1 for all t>0and 0<x<1

3. Prove the comparison principle for the diffusion equation : If u and v are two solutions, and if $u \le v$ for t=0, for x=0, and for x=*l* then $u \le v$ for $0 \le t < \infty, 0 \le x \le l$

Check your progress

Prove the maximum principal.

3.5 DIFFUSION ON THE WHOLE LINE

Our purpose in this section is to solve the problem

As with the wave equation, the problem on the infinite line has a certain "Purity", which makes it easier to solve than the finite-interval problem. (The effects of boundaries will be discussed in the next several chapters.) Also as be served by a method very different from the methods used before.(The play no major role in the analysis.) Because the solution of (1) is not easy to derive, we first set the stage by making some general comments.

Notes

Our method is to solve it for a particular and then build the general solution from this particular one. We'll use five basic invariance properties of the diffusion equation (1).

(a) The translate u(x – y,t) of any solution u(x,t) is another solution for any fixed y.

(b) Any derivative of a solution is again a solution.

(c) A linear combination of solutions of (1) is again a solution of (1)(This is just linearly.)

(d) An integral of solutions is again a solution. Thus if S(x,t) is a solution of (1), then so is S(x – y,t) and so is

For any function g(y), as long as this improper integral converges approximately.

(We"ll worry about convergence later.) In fact, (d) is just a limiting form of (c).

(e) If u(x,t) is a solution of (1), so is the dilated function , for any a > 0. Prove this by the chain rule: Let v(x,t)= .

Then and

Our goal is to find a particular solution of (1) and then to construct all the other solutions using property (d). The particular solution we will look for is the one, denoted , which satisfies the special initial condition

The reason for this choice is that this initial condition does not change under dilation. We'll find in steps.

STEP 1: WE'LL LOOK FOR OF THE SPECIAL FORM

$$Q(x,t) = g(p)\, where\ p = \frac{x}{\sqrt{4kt}} \qquad (4)$$

And g is a function of only one variable (to be determined). (The $\sqrt{4k}$ factor is included only to simplify a later formula.)

Why do we expect Q to have this special form? Because property (e) says that equation (1) doesn't "see" the dilation

$x \to \sqrt{ax}, t \to at.$ Clearly. (3) doesn't change at all under the dilation. So Q(x,t), which is defined by conditions (1) and (3). Ought not seethe dilation either. How could that happen In only one way: if Q depends on x and t solely through the combination $x/\sqrt{t}.$ For the dilation takes $x/\sqrt{t}$ into $\sqrt{ax/}\sqrt{at} = x/\sqrt{t}.$ Thus let $p = x/\sqrt{4kt}$ and lok for Q which satisfies (1) and (3) and has the form (4).

Step 2 : Using (4), we convert (1) into an ODE for g y use of the chain rule:

$$Q_t = \frac{4g}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4lt}}g'(P)$$

$$Q_x = \frac{4g}{dp}\frac{\partial p}{\partial t} = \frac{1}{\sqrt{4kt}}g'(P)$$

$$Q_{xx} = \frac{4Q_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4lt}}g'(P)$$

$$0 = Q_t - kQ_{xx} = \frac{1}{t}\left[-\frac{1}{2}pg^{'}(p) - \frac{1}{4}g^{''}(p)\right]$$

Thus

$$g^{''} + 2pg^{'} = 0.$$

This ODE is easily solved using the integrating actor exp

$\int 2pdp = xp(p^2).We\, getg^{'}(p) = c_1 \exp(-p^2) and$

$$Q(x,t) = g(p) = c_t \int e^{-p^2} dp + c_2$$

<u>Step -3</u> We find a completely explicit formula for Q. We 've just shown that

$$Q(x,t) = c_t \int_0^{x/\sqrt{4kt}} e^{-o2} dp \_ c_{2.}$$

This formula is valid only for t>0. Now use (3), expressed as a limit as follows

$$Q(x,t) = c_1 \int_0^{x/\sqrt{44kt}} e^{-p^2} dp + c_2.$$

Notes

<u>Step 4:</u> Having found Q, we now define $S = \partial Q / \partial x.$ (The explicit formula for S will be written below.) By property (b),S is also a solution of (1). Given any function ϕ ,we also define

By property (d), u is another solution of (1), we claim that u is the unique solution of (1),(2). To verify the validity of (2),w e write.

$$If x > 0, 1 = \lim_{f\square 0} Q = c_1 \int_0^{-\infty} e^{-p^2} dp + c_2 = c_1 \frac{\sqrt{\pi}}{2} + c_2.$$

$$If \; x < 0, 0 = \lim_{f\square 0} Q = c_1 \int_0^{-\infty} e^{-p^2} dp + c_2 = -c_1 \frac{\sqrt{\pi}}{2} + c_2.$$

See Exercise . Here lim means limit from the right. This determines the coefficients $c_1 = 1\sqrt{\pi} and \; c_2 = \frac{1}{2}$. Therefore, Q is the function

$$\boxed{Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2} dp}$$

By property (d), u is another solution of (1). We claim that u is the unique solution of (1),(2). To verify the validity of (2),we write

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)dy$$

$$-\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)\phi(y)dy$$

$$= +\int_{-\infty}^{\infty} \frac{\partial}{\partial y} Q(x-y,t)\phi^1(y)dy - Q(x-y),t)\phi(y)\Big|_{y=-\infty}^{y=+\infty}$$

Upon integrating by parts. We assume these limits vanish. In particular. Let's temporarily assume that $\phi(y)$ I self equals zero for $|y|$ large. Therefore.

$$u(x,0) = \int_{-\infty}^{-\infty} Q(x-y,0)\phi'(y)dy$$

$$= \int_{-\infty}^{x} \phi'(y)dy = \phi\Big|_{-\infty}^{x} = \phi(x)$$

Because of the initial condition for Q and the assumption that

$$u(x,0)=\int_{-\infty}^{\infty}Q(x-y,0)\phi'(y)dy$$

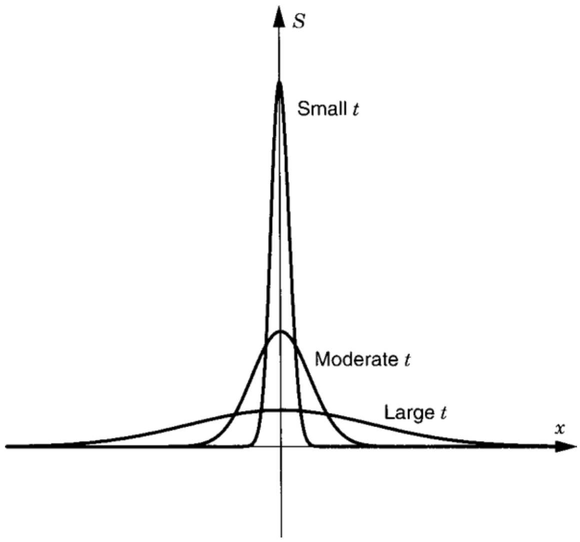
$$=+\int_{-\infty}^{\infty}Q(x-y,t)\phi'(y)dy-Q(x-y,t)\phi(y)\Big|_{-\infty}^{x}=\phi(x)$$

Because of the initial condition for Q and the assumption that $\phi(-\infty)=0.$ This is the initial condition (2). We conclude that 96) is our solution formula, where

$$\boxed{S=\frac{\partial Q}{\partial x}==\frac{1}{2\sqrt{\pi kt}}e^{-x^2/4kt}\, for\, t>0.} \qquad (7)$$

That is,

$$\boxed{u(x,t)=\frac{1}{4\sqrt{\pi kt}}\int_{-\infty}^{\infty}e^{-x-y)^2}} \qquad (8)$$



S(x,t) is known as the source function, Green's function, fundamental solution, Gaussian, or propagator of the diffusion equation, or simply the diffusion kernel. It gives the solution of (1),(2) with any initial datum ϕ. He formula only gives the solution for t>0. When t=0 it makes no sense.

The source function S(x,t) is defined for all real x and for all t>0. S(x,t) is positive and is even in x[S(-x,t)=S(x,t)]. It looks like figure 1 for various values of t. For large t, it is very spread out. For small, t, it is a

Notes

very tall thin spike (a"deha function") of height $(4\pi kt)^{-1/2.}$ The area under its graph is

$$\int_{-\infty}^{\infty} S(x,t)dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-q^2} dq = 1$$

By substituting math type .

Now look more carefully at the sketch of 5(x,t) for a very small t, If we cut out the tall spike, he rest of S(x,t) is very small. Thus.

$$\max_{|x| > \partial} S(x,t) \to 0 \ as \ \ t \to 0$$

Notice that the value of the solution u(x,t) given y (6) is a kind of weighted average of the initial values around the point x., Indeed, we an write

$$u(x,t) = \int_{\alpha}^{\alpha} S(x-y,t)\phi(y)dy \,\square\, \sum_{t} S(x-y,t)\phi(y)\Delta y_i$$

Approximately. This is the average of the solutions S(x-y T) with the weights For very small t, the source function is a spike so that the formula exaggerates the values of near x. For any t>0 the solution is a spread out version of the initial values at t=0.

Here's the physical interpretation. Consider diffusion . S(x-y,t) represents the result of a unit mass (say, I gram) of substance located at time zero exactly at the position y which is diffusing (spreading out) as time advances. For any initial distribution of concentration, the amount of substance initially in the interval y spreads out in time and contributes approximately the term All these contributions are added up to get the whole distribution of matter. Now consider heat flow. S(x-y,t)represents the result of a "hot spot" at y at time 0. The hot spot is cooling off and spreading its heat along the rod.

Another physical interpretation is Brownian motion, where particles move randomly in space. For simplicity, we assume that the motion is one dimensional; that is, the particles move along a tube. Then the probability that a particle which begins at position x ends up in the interval (a,b) at time t is precisely for some constant k, where S is define in (7). In other words, if we let u(x,t) be the probability density (Probability density is (x), then the probability at all later times is given by formula (6). That is, u(x,t) satisfies the diffusion equation.

It is usually impossible to evaluate integral (8) completely in terms of elementary functions . Answers to particular problems, that is, to articular initial data ,are sometimes expressible in terms of the error function of statistics.

$$\varepsilon rf\,(x) - \frac{2}{\sqrt{\pi}} \int_0^x e^{-2} dp.$$

Notice that by Exercise 6, lim

<u>Example 1</u>

From (5) we can write Q(x,t) in terms of as

$$Q(x,t) = \frac{1}{2} + \frac{1}{2}\varepsilon rf\left(\frac{x}{\sqrt{4kt}}\right)$$

<u>Example -2 :</u>Solve the diffusion equation with the initial condition u(x,0) To do so, we simply plug this into the general formula (8):

$$u(x,t) = \frac{x}{\sqrt{4kt}} \int_{-\alpha}^{\alpha} e^{-(x-y)^2/4kt} e^{-y} dy.$$

This is one of the few fortunate examples that can be integrated. The exponent is

$$-\frac{x^2 - 2xy + y^2 + 4kty}{4kt}$$

Completing the square in the y variable, it is

$$-\frac{y^2 - 2xy + y^2 + 4kty}{4kt}$$

We let

So that then.

$$-\frac{(y + 2kt - x)^2}{4kt} + kt - x.$$

We let $p = (y + 2kt - x)/\sqrt{4kt} so\, that\, dp - dy/\sqrt{4kt}. Then$

Notes

$$u(x,t) = e^{kt-x}\int_{-\infty}^{\infty} e^{-p^2}\,\frac{dp}{\sqrt{\pi}}\,e^{kt-x}$$

By the maximum principle, a solution in a bounded interval can not grow in time. However, this particular solution grows, rather than decays, in time. The reason is that the left side of the rod is initially very hot math type

And the heat gradually diffuses throughout the rod.

EXERCISE:

1. Solve the diffusion equation with the initial condition

$\phi(x) = l \;\; for\,|x| < l$ and $\phi(x) = 0 \;for\,|x| > l$

Write your answer in terms of $\xi rf\,(x)$

2. Do the same for $\phi(x) = 1 \;for\; x > 0 \,and\, \phi(x) = 3 \;for\; x < 0$

3. Solve the diffusion equation if

$\phi(x) = e^{-x} \;\; for\; x > 0 \,and \phi(x) is \;\; for\; x < 0$

Check your progress

1. Explain about five basic invariance properties of the diffusion equation.

3.6 COMPARISION OF WAVES AND DIFFUSION

We have seen that the basic property of waves is that information gets transported in both directions at a finite speed the basic property of diffusions is that the initial disturbance gets spread out in a smooth fashion and gradually disappears. The fundamental properties of these two equations can be summarized in the following table.

Property	Waves	Diffusions
(i) Speed of propagation ? (ii) Singularities	Finite Transported along characteristic	Infinite Lost immediately

for t>0?	(speed =c)	Yes (at least for
(iii) Well-posed for	Yes	bounded solutions)
t>0?	Yes	No
(iv) Well-posed for	No	Yes
t<0?	Energy is	Decays to zero (if
(v) Maximum	Constant so does	ϕ integrable)
principle	not decay	Lost gradually
(vi) Behavior as	Transported	
(vii) Information		

For the wave equation we have seen most of these properties already. That there is no maximum principle is easy to see. Generally speaking, the wave equation just moves information along the characteristic lines. In more than one dimension we'll see that it spreads information in expanding circles or spheres.

For the diffusion equation we discuss property (ii), hat singularities are immediately lost, in future sections.

The solution is differentiable to all orders even if the initial data are not Properties (iii),(v), and (vi) have been shown already. The fact that information is gradually lost [property (vii)] is clear from the graph of a typical solution, for instance, from S(x,t).

As for property (i) for diffusion equation, notice from formula (2,4,8) that the value of u(x,t) depends on the values of the initial datum $\phi(y)$ for all y, were $-\infty < y < \infty$. Conversely, the value of ϕ at a point x_0 has an immediate effect everywhere (fort>0),even though most of its effect is Exercise 2(b) shows that solutions of the diffusion equation can raved at any speed. This is in stark contrast to the wave equation (and all hyperbolic equations).

As for (iv), there are several ways to see that the diffusion equation is not well-posed for t<0)("backward in time").One way is the following. Let

$$u(x,t) = \frac{1}{n}\sin nxe^{-n^2kt}$$

Notes

You can check hat this satisfies the diffusion equation for all x,t.Also

$u_n(x,0) = n^{-1}\sin nx \to 0$ Uniformly as $n \to \infty$. But consider any t<0, say t-1. Then

$t = -1. Then\, u_n(x,-1) = n^{-1}\sin nxe^{+kn^2}$ Uniformly as $n \to \infty$. except for a few x. Thus u_n

is close to the zero solution at the t=0 but not at time t=-1. This violates the stability, in the uniform sense at least.

Another way is to let $u(x,t) = S(x,t+1)$. This is a solution of the diffusion equation

$u_1 = ku_{xx}\, for\, t < -1, -\infty < x < \infty$. But $u(0,t) \to \infty as\, t \;\; -1$, as we saw above. So we cannot solve backwards in time with the perfectly nice-looking initial data

In time with the perfectly nice-looking initial data $(4\pi k)^{-1}e^{-x^{2/4.}}$

Besides, any physicist knows that heat flow, Brownian motion, and so on, are irreversible professes Going backward leads to chaos.

Exercise:

1.Show that there is no maximum principal for the wave equation.

2. Consider the travelling wave u(x, t)=f(x-at) where f is a given function of one variable.

a) If it is the solution of the wave equation, show that the speed must be $a = \pm c$ (unless f is a linear function)

b) If it is the solution of the diffusion equation, find f and show that the speed a is arbitrary.

Check your progress

3. Write the comparisions of waves and diffusion

3.7 LET US SUM UP

In this unit we have discussed the Wave equation, Causality and energy, the diffusion equation, Different ion on the whole line, Comparison of waves and diffusion and solved examples. Uses of PDEs on physical situations . The most fundamental properties of the PDEs can be found most easily without the complications of boundary conditions. Principle of causality. In electromagnetic theory the equations are Maxwell's. One-dimensional diffusion equation. Maximum principle.

3.8 KEY WORDS

1. The wave equation is $u_{tt} = c^2 u_{xx}\ for -\infty < x < +\infty$
2. The effect of an initial position $\phi(x)$ is a pair of waves traveling in either direction at speed c and at half the original amplitude.
3. An initial condition (position or velocity or both) at the point $(x_0, 0)$ can affect the solution fo t>0 only in the shaded sector, which is called the domain of influence of the point $(x_0, 0)$.
4. Kinetic energy is $\frac{1}{2}mv^2$, which in our case takes the form $KE \frac{1}{2} p \int u_t^2 dx.$
5. Diffusions are very different from waves, and this is reflected in the mathematical properties of the equations.
6. Maximum Principle: If $u(x,t)$ satisfies the diffusion equation in a rectangle $(say, o \le x \le l, 0 \le t \le T)$ in space-time, then the maximum value of $u(x,t)$ is assumed either initially (t=0) or on the lateral sides $(x = 0\ or\ x = l)$
7. The maximum principle can be used to give a proof of uniqueness for the Dirichlet problem for the diffusion equation.
8. The basic property of waves is that information gets transported in both directions at a finite speed.

3.9 QUESTIONS FOR REVIEW

1. Discuss the wave equation

2. Discuss the diffusion equation

3. Write comparisons of waves and diffusion.

3.10 SUGGESTED READINGS AND REFERENCES

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11. Partial Differential Equations,-*Victor Ivrii*

3.11ANSWERS TO CHECK YOUR PROGRESS

1. See section 3.5
2 .See section 3.5

3. See section 3.6

UNIT-4 REFLECTIONS AND SOURCES

STRUTURE

4.0 OBJECTIVE

After studying this unit we will learn and understand about Diffusion of the half –line, Reflection of waves, Diffusion with a source, Source on a half line, Waves with a source, Well-posedness, Method of using Green’s theorem.

4.1 INTRODUCTION

In this chapter we solve the simplest reflection problems, when there is only a single point of reflection at one end of a semi-infinite line. In Chapter 4we shall begin a systematic study of more complicated reflection problems. In Sections 4.5 and 4.6 we solve problems with sources: that is, the inhomogeneous wave and diffusion equations.

4.2 DIFFUSION ON THE HALF-LINE

Let's take the domain to be D = the half-line $(0,\infty)$ and take the Dirichlet

boundary condition at the single endpoint x = 0. So the problem is

$$\upsilon_t - k\upsilon_{xx} = 0 \quad in\{0 < x < \infty,\ 0 < t < \infty\},$$
$$\upsilon(x,0) = \phi(x) \quad for\, t = 0$$
$$\upsilon(0,t) = 0 \quad for\ x = 0$$

The PDE is supposed to be satisfied in the open region {0 <x <∞,0 <t <∞}.

If it exists, we know that the solution v(x, t) of this problem

is unique because of our discussion in the previous section.

It can be interpreted, for instance, as the temperature in a very long rod with one end immersed in a reservoir of temperature zero and with insulated sides.

In fact, we shall reduce our new problem to our old one. Our method uses the idea of an odd function. Any function $< 1.\frac{\in}{2} = \frac{\in}{2}.$ that satisfies $\psi(-x) \equiv -\psi(+x)$ is called an odd function. Its graph $y = \psi(x)$ is symmetric with respect to the origin

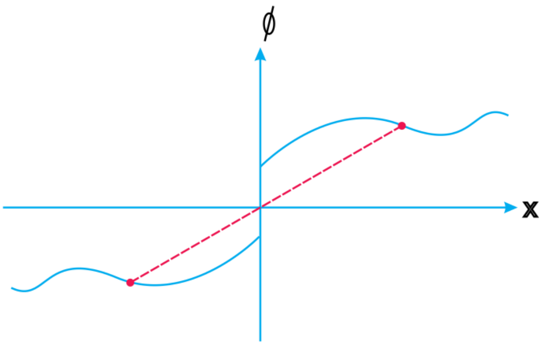


Figure 1

(See Figure 1).Automatically (by putting $x=0$ in the definition), $\psi(0)=0$ .For a detailed discussion of odd and even functions, see section 5.2.

Now the initial datum $\phi(x)$ of our problems is defined only for $x \geq 0$.

Let ϕ_{odd} be the unique odd extension of ϕ to the whole line. That is,

$$\phi_{odd}(x)=\begin{cases}\phi(x) & for\, x>0\\ -\phi(-x) & for\, x<0\\ 0 & for\, x=0.\end{cases}$$

The extension concept too is discussed in the coming chapter.

Let $u(x,t)$ be the solution of

$$u_t - ku_{xx} = 0$$
$$u(x,0)=\phi_{odd}(x)$$

For the whole line $-\infty < x < \infty, 0 < t < \infty.$

We know that, $u(x,t)=\int_{-\infty}^{\infty} S(x-y,t)\phi_{odd}(y)dy.$

Its "restriction,"

$$\upsilon(x,t)=u(x,t) \quad for\, x>o,$$

Will be the unique solution of our new problem (1) . There is no difference at all between υ and u except that the negative values of x are not considered when discussing υ .

Why is $\upsilon(x,t)$ the solution of (1)? Notice first that $u(x,t)$ must also be an odd function of x .

That is, $u(-x,t)=-u(x,t)$. Putting $\mathrm{x}=0$, it is clear that $\mathrm{u}(0,t)=0$ .So the boundary condition $\upsilon(0,t)=0$ is automatically satisfied! Furthermore, υ solves the PDE as well as the initial condition for $x>0$, simply because it is equal to u for $x>0$ and u satisfies the same PDE for all x and the same initial condition for $x>0$.

The explicit formula for $\upsilon(\mathrm{x},t)$ is easily deduced from (4) and (5). From (4) and (2) we have

$$u(x,t)=\int_0^\infty S(x-y,t)\phi(y)\,dy-\int_{-\infty}^0 S(x-y,t)\phi(-y)\,dy.$$

Changing the variable –y to +y in the second integral, we get

$$u(x,t)=\int_0^\infty [S(x-y,t)-S(x+y,t)]\phi(y)\,dy.$$

(Notice the change in the limits of integration.) Hence for $0<x<\infty, 0<t<\infty$ we have

$$\upsilon(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_0^\infty \left[e^{-(x-y)^2/4kt}-e^{-(x+y)^2/4kt}\right]\phi(y)\,dy.$$

This is the complete solution formula for (1).

We have just carried out the method of odd extensions or reflection method, so called because the graph of $\phi(x)$ across the origin.

Example 1:

Solve (1) with $\phi(x)\equiv 1$. The solution is given by formula (6). This case can be simplified as follows. Let $p=(x-y)/\sqrt{4kt}$ in the first integral and $q=(x+y)/\sqrt{4kt}$ in the second integral. Then

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^{x/\sqrt{4kt}} e^{-p^2 dp/\sqrt{\pi}} - \int_{x/\sqrt{4kt}}^{+\infty} e^{-q^2 dq/\sqrt{\pi}} \\
&= \left[\tfrac{1}{2} + \tfrac{1}{2}\xi rf\left(\tfrac{x}{\sqrt{4kt}}\right)\right] - \left[\tfrac{1}{2} - \tfrac{1}{2}\xi rf\left(\tfrac{x}{\sqrt{4kt}}\right)\right] \\
&= \xi rf\left(\tfrac{x}{\sqrt{4kt}}\right).
\end{aligned}
$$

Now let’s play the same game with the Neumann problem

$$
\begin{aligned}
&w_t - kw_{xx} = 0 \ for\ 0 < x < \infty, 0 < t < \infty \\
&w(x,0) = \phi(x) \\
&w_x(0,t) = 0.
\end{aligned}
$$

In this case the reflection method is to use even, rather than odd, extensions. An even function is a function ψ such that $\psi(-x) = +\psi(x)$. If ψ is an even function, then differentiation shows that its derivative is an odd function. So automatically its slope at the origin is zero: $\psi'(0) = 0$. If $\phi(x)$ is defined only on the half-line, its even extension is defined to be

$$
\begin{aligned}
\phi_{even}(x) = \{&\phi(x) \quad for\ x \ge 0 \\
\{&+\phi(-x)\ for\ x \le 0
\end{aligned}
$$

By the same reasoning as we used above, we end up explicit formula for $w(x,t)$. It is

$$w(x,t) = \tfrac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}\right]\phi(y)\,dy.$$

This is carried out in Exercise 3. Notice that the only difference between (6) and (9) is a single minus sign!

EXERCISE:

1. $Solve\ u_t = ku_{xx}; u(x,0) = e^{-x}; u(0,t) = 0\ on\ the\ half-line\ 0 < x < \infty.$
2. $Solve\ u_t = ku_{xx}; u(x,0); u(0,t) = 1\ on\ the\ half-line\ 0 < x < \infty.$

3. Derive the solution formula for the half-line Neumann Problem $w_t - kw_{xx} = 0\ for\ 0 < x < \infty, 0 < t < \infty; w_x(0,t) = 0; w(x,0) = \phi(x).$

4. (a) Use the method of Exercise 4 to solve the Robin Problem:

$$DE: u_t = ku_{xx} \qquad on\,the\,half-line\, 0 < x < \infty$$
$$(and\, 0 < t < \infty)$$
$$IC: u(x,0) = x \qquad for\, t = 0\, and\, 0, x < \infty$$
$$BC: u_x(0,t) - hu(0,t) = 0 \qquad for\, x = 0,$$

Where h is a constant.

(b) Generalize the method to the case of general initial data $\phi(x)$.

4.3 REFLECTIONS OF WAVES

Now we try the same kind of problems for wave equation as we did in Section 3.1 for the diffusion equation. We again begin with the Dirichlet problem on the half-line $(0,\infty)$. Thus the problem is

$$DE: u_{tt} - c^2\upsilon_{xx} = 0 \qquad for\, 0 < x < \infty$$
$$and\, -\infty < t < \infty$$
$$IC: \upsilon(x,0) = \phi(x), \upsilon_t(x,0) = \psi(x) \qquad for\, t = 0$$
$$and\, 0 < x < \infty$$
$$BC: \upsilon(0,t) = 0 \qquad for\, x = 0$$
$$and\, -\infty < t < \infty.$$

The reflection method is carried out in the same way as in section 3.1. Consider the odd extension of both of the initial functions to the whole line, $\phi_{odd}(x)\, and\, \psi_{odd}(x)$. Let $\mathrm{u}(x,t)$ be the solution of the initial-value problem on $(-\infty,\infty)$ with the initial data $\phi_{odd}(x)\, and\, \psi_{odd}$. Then $\mathrm{u}(x,t)$ is once again an odd function of x . Therefore, $\mathrm{u}(0,t) = 0,$ so that the boundary condition is satisfied automatically. Define $\upsilon(x,t) = u(x,t)\, for\, 0 < x < \infty$ [the restriction of u to the half-line]. Then $\upsilon(x,t)$ is precisely the solution we are looking for. From the formula in Section 2.1, we have for $x \geq 0,$

$$\upsilon(x,t) = u(x,t) = \frac{1}{2}\left[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{odd}(y)\,dy.$$

Let's "unwind" this formula, recalling the meaning of the odd extensions. First we notice that for $x > c|t|$ only positive arguments occur in the formula,

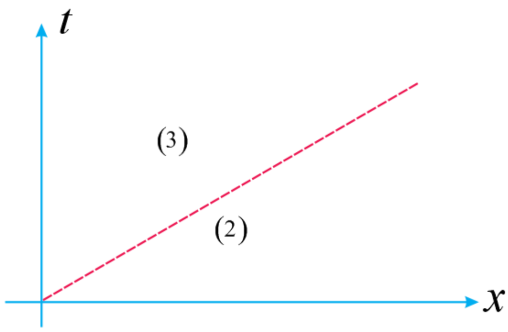


Figure 1

So that $u(x,t)$ is given by the usual formula:

$$\upsilon(x,t) = \tfrac{1}{2}\left[\phi(x+ct) + \phi(x-ct) + \tfrac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy\right]$$
$$for\; x > c|t|.$$

But in the other region $0\,x > c|t|$, we have $\phi_{odd}(x-ct) = -\phi(ct-x)$, and so on, so that

$$\upsilon(x,t) = \tfrac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \tfrac{1}{2c}\int_{0}^{x+ct} \psi(y)\,dy + \tfrac{1}{2c}\int_{x-ct}^{0} \left[-\psi(-y)\right]dy.$$

Notice the switch in signs! In the last term we change variable $y \to -y$ to get $\tfrac{1}{2c}\int_{cl-x}^{cl+x} \psi(y)\,dy.$ Therefore,

$$\upsilon(x,t) = \tfrac{1}{2}\left[\phi(ct+x) - \phi(ct-x)\right] + \tfrac{1}{2c}\int_{cl-x}^{cl+x} \psi(y)\,dy$$

For $0 < x < c|t|$.The complete solution is given by the pair of formulas (2) and (3). The two regions are sketched in Figure 1 for $t > 0$.

Graphically, the result can be interpreted as follows. Draw the backward characteristics from the point (x,t). In case (x,t) is in the region x<ct, one of the characteristics hits the t axis $(x = 0)$ before it

Notes

hits the x axis, as indicated in Figure 2. The formula (3) shows that the reflection induces a change of

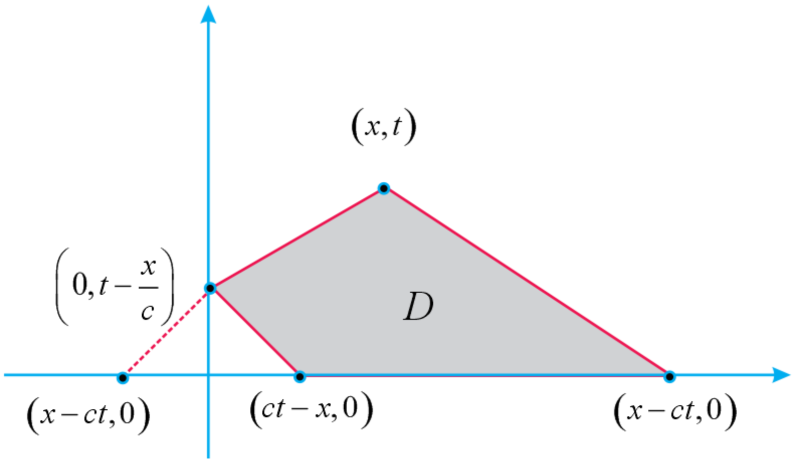


Figure-2

Sign. The value of $\upsilon(x,\mathrm{t})$ now depends on the values of ϕ at the pair of points $ct \pm x$ and on the values of ψ have canceled out. The shaded area D in Figure 2 is called the domain of dependence of the point (x,t).

The case of the Neumann problem is left as an exercise.

THE FINITE INTERVAL

Now let’s consider the guitar string with fixed ends:

$$\upsilon_{tt} = c^2\upsilon_{xx} \quad \upsilon(x,0)=\phi(x) \quad \upsilon_t(x,0)=\psi(x) \quad for \ 0<x<l, \upsilon(0,t)=\upsilon(l,t)=0.$$

This problem is much more difficult because a typical wave will bounce back and forth an infinite number of times. Nevertheless, let’s use the method of reflection. This is a bit tricky, so you are invited to skip the rest of this section if you wish.

This initial data $\phi(x) \, and \, \psi(x)$ are now given only for $0<x<l.$ We extend them to the whole line to be “odd” with respect to both x=0 and x=l:

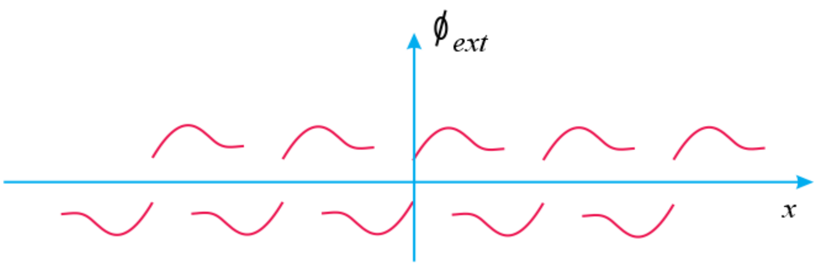
$$\phi_{ext}(-x) = -\phi_{ext}(x) \quad and \quad \phi_{ext}(2l-x) = -\phi_{ext}(x).$$

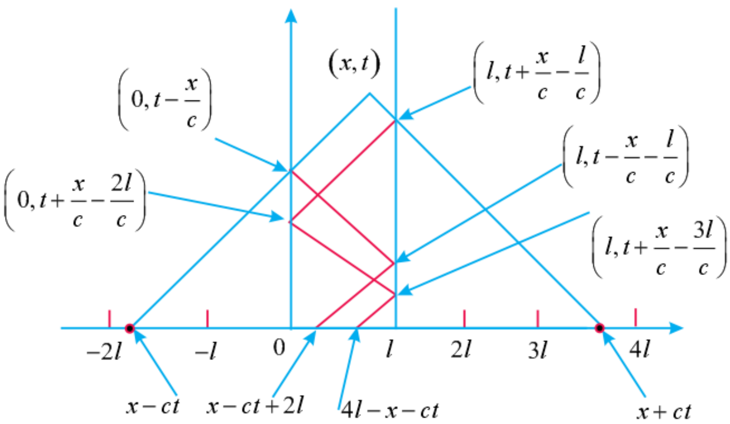
The simplest way to do this is to define

$$\phi_{ext}(x) = \begin{cases} \phi(x) & for \quad 0 < x < 1 \\ -\phi(-x) & for \quad -l < x < 0 \\ \textbf{extended to be of period 2l.} \end{cases}$$

See Figure 3 for an example. And see Section 5.2 for further discussion. "Period 2l" means that $\phi_{ext}(x+2l) = \phi_{ext}(x)$ for all x. We do exactly the same for $\psi(x)$(defined for $0 < x < 1$) to get $\psi_{ext}(x)$ defined for $-\infty < x < \infty$.

Now let $u(x,t)$ be the solution of the infinite line problem with the extended initial data. Let υ be the restriction of u to he interval $(0l)$. Thus $\upsilon(x,t)$ is





Given by the formula

$$\upsilon(x,t) = \tfrac{1}{2}\phi_{ext}(x+ct) + \tfrac{1}{2}\phi_{ext}(x-ct) + \tfrac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(s)\,ds$$

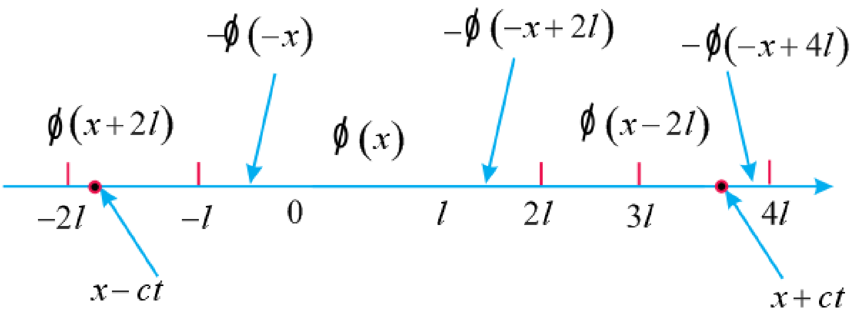
for $0 \le x \le l$. This simple formula contains all the information we need. But to see it explicitly we must unwind the definitions of ϕ_{ext} and ψ_{ext}. This will give a resulting formula which appears quite complicated because it includes a precise description of all the reflections of the wave at both of the boundary points $x = o$ and $x = l$.

The way to understand the explicit result we are about to get is by drawing a space-time diagram (Figure 4). From the point (x,t), we draw the two characteristic lines and reflect them each time they hit the boundary. We keep track of the change of sign at each reflection. We illustrate the result in Figure 4 for the case of a typical point. (x,t).

We also illustrate in Figure 5 the definition of the extended function $\phi_{ext}(x)$.(The same picture is valid for ψ_{ext}.) For instance, for the point (x,t).as drawn in Figure 4 and 5, we have

$$\phi_{ext}(x+ct) = -\phi(4l-x-ct)\, and\ \phi_{ext}(x-ct) = +\phi(x-ct+2l).$$

The minus coefficient on $-\phi(-x-ct+4l)$ comes from the odd number of reflections $(=3)$. The plus coefficient on $+\phi(x-ct+2l)$ comes from the even



number of reflections $(=2)$. Therefore, the general formula (5) reduces to

$$\begin{aligned} \upsilon(x,t) = &\frac{1}{2}\phi(x-ct+2l) - \frac{1}{2}\phi(4l-x-ct) \\ &+\frac{1}{2c}\Big[\int_{x-ct}^{-l} \psi(y+2l)\,dy + \int_{-l}^{0} -\psi(-y)\,dy \\ &+\int_{0}^{1} \psi(y-2l)\,dy + \int_{3l}^{x+ct} -\psi(-y+4l)\,dy\ \Big] \end{aligned}$$

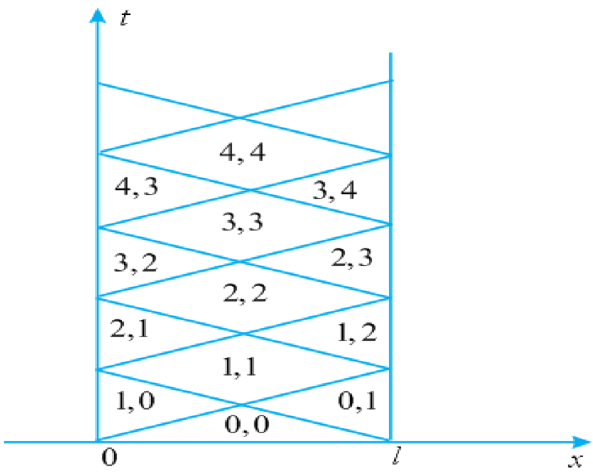
But notice that there is an exact cancellation of the four middle integrals, as we see by changing $y \to -y\, and\ y-2l \to -y+2l$. So, changing variables in the two remaining integrals, the formula simplifies to

$$\upsilon(x,t) = \frac{1}{2}\phi(x-ct=2l) - \frac{1}{2}\phi(4l-x-ct) + \frac{1}{2c}\int_{x-ct+2l}^{l} \psi(s)\,ds + \frac{1}{2c}\int_{l}^{4l-x-ct} \psi(s)\,ds.$$

Therefore, we end up with the formula

$$\upsilon(x,t) = \frac{1}{2}\phi(x-ct+2l) - \frac{1}{2}\phi(4l-x-ct) + \int_{x-ct+2l}^{4l-x-ct} \psi(s)\frac{ds}{2c}$$

at the point (x,t) illustrated, which has three reflections on one end and two on the other. Formula (6) is valid only for such points.



The solution formula at any other point (x,t) is characterized by the number of reflections at each end $(x=0,l)$. This divides the space-time picture into diamond-shaped regions as illustrated in Figure 6. With each diamond the solution $\upsilon(x,t)$ is given by a different formula. Further examples may be found in the exercises.

The formulas explain in detail how the solution looks. However, the method is impossible to generalize to two- or three-dimensional problems, nor does it work for the diffusion equation at all. Also it is very complicated! Therefore, in Chapter 4 we shall introduce a completely different method (Fourier's) for solving problems on a finite interval.

EXERCISE:

Notes

1. Solve the Neumann problems for the wave equation on the half-line $0 < x < \infty$.

The longitudinal vibrations of a semi-infinite flexible rod satisfy the wave equation $u_{tt} = c^2 u_{xx}$ *for* $x > 0$. Assume that the end $x = 0$ is free $(u_{tt} = 0)$; it is initially at rest but has a constant initial velocity V for $a < x < 2a$ and has zero initial velocity elsewhere. Plot u versus x at the times $t = 0, a/c, 3a/2c, 2a/c$, and $3a/c$.

A wave $f(x+ct)$ travels along semi-infinite string $(0 < x < \infty)$ for $t < 0$. Find the vibrations $u(x,t)$ of the string for $t > 0$ if the end $x = 0$ is fixed.

2. Solve $u_{tt} = 4u_{xx}$ *for* $0 < x < \infty, u(0,t) = 0, u(x,0) \equiv 1, u_t(x,0) \equiv 0$ using the reflection method. This solution has a singularity; find its location.
3. Solve

$$u_{tt} = c^2 u_{xx} \text{ in } 0 < x < \infty, 0 \le t < \infty, u(x,0) = 0, u_t(x,0) = V, u_t(0,t) + a u_x(0,t) = 0,$$

Where V, a, and c are positive constants and $a > c$.

(a) Show that $\phi_{odd}(x) = (sign\, x)\phi(|x|)$.

(b) Show that $\phi_{ext}(x) = \phi_{odd}(x - 2l[x/2l])$, where $[.]$ denotes the greatest integer function.

(c) Show that

$$\phi_{ext}(x) = \begin{cases} \phi\left(x - \left[\frac{x}{l}\right]l\right) & \text{if } \left[\frac{x}{l}\right] \text{even} \\ -\phi\left(-x - \left[\frac{x}{l}\right]l - l\right) & \text{if } \left[\frac{x}{l}\right] \text{odd}. \end{cases}$$

4. For the wave equation in a finite interval $(0,l)$ with Dirichlet conditions, explain the solution formula within each diamond-shaped region.

(a) Find

$u\left(\frac{2}{3},2\right) if\ u_{tt}=u_{xx}\ in\ 0<x<1, u(x,0)=x^2(1-x), u_t(x,0)=(1-x^2), u(0,t)=u(1,t)=0.$

(b) Find $u\left(\frac{1}{4},\frac{7}{2}\right)$.

Check your progress

1. Explain about Reflection of waves

4.4 DIFFUSION WITH A SOURCE

In this section we solve the in homogeneous diffusion equation on the whole line,

$$u_t-ku_{xx}=f(x,t)\quad(-\infty<x<\infty, 0<t<\infty)$$
$$u(x,0)=\phi(x)$$

With $f(x,t)$ and $\phi(x)$ arbitrary given functions. For instance, if $u(x,t)$ represents the temperature of a rod, then $\phi(x)$ is the initial temperature distribution and $f(x,t)$ is a source

(or sink) of heat provided to the rod at later times.

We will show that the solution of (1) is

$$u(x,y)=\int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$$
$$+\int_0^t\int_{-\infty}^{\infty} S(x-y,t-s)\,f(y,s)\,dy\,ds.$$

Notice that there is the usual term involving the initial data ϕ and another term involving the source f. Both terms involve the source function s.

Notes

Let's begin by explaining where (2) comes from. Later we will actually prove the validity of the formula. (If a strictly mathematical proof is satisfactory to you, this paragraph and the next two can be skipped.) our explanation is an analogy. The simplest analogy is the ODE.

$$\frac{du}{dt} + Au(t) = f(t), \quad u(0) = \phi.$$

Where A is a constant. Using the integrating factor e^{tA}, the solution is

$$u(t) = e^{-tA}\phi + \int_0^t e^{(s-t)A} f(s)\,ds.$$

A more elaborate analogy is the following. Let's suppose that ϕ us an n-vector, $u(t)$ is an n-vector function of time, and A is a fixed $\mathrm{n}\times\mathrm{n}$ matrix.

Then (3) is a coupled system of n linear ODEs. In case $f(t) \equiv 0$, the solution of (3) is given as $u(t) = S(t)\phi$, where $S(t)$ is the matrix $S(t) = e^{-tA}$. So in case $f(t) \neq 0$, an integrating factor for (3) is $S(-t) = e^{-tA}$. So in case $f(t) \neq 0$, an integrating factor for (3) is $S(t) = e^{-tA}$. Now we multiply (3) on the left by this integrating factor to get

$$\frac{d}{dt}\left[S(-t)u(t)\right] = S(-t)Au(t) = S(-t)f(t).$$

Integrating from 0 to t, we get

$$S(-t)u(t) - \phi = \int_0^t S(-s)f(s)\,ds.$$

Multiplying this by $\mathrm{S}(t)$, we end up with the solution formula

$$u(t) = S(t)\phi = \int_0^t S(t-s)f(s)\,ds.$$

The first term in (5) represents the solution of the homogeneous equation, the second the effect of the source $f(t)$. For a single equation, of course, (5) reduces to (4).

Now let's return to the original diffusion problem (1). There is an analogy between (2) and (5) which we now explain. The solution of (1) will have two terms. The first one will be the solution of the homogeneous problem, already solved in Section 2.4, namely

$$\int_{-\infty}^{\infty} S(x-y,t)\phi 9 y dy = (\varphi(t)\phi)(x).$$

$S(x-y,t)$ is the source function given by the formula (2.4.7) Here we are using $\varphi(t)$ to denote the source operator, which transforms any function ϕ to the new function given by the integral in (6). (Remember: Operators transform functions into functions.) We can now guess what the whole solution to (1) must be. In analogy to formula (5), We guess that the solution of (1) is

$$u(t) = \varphi(t)\phi + \int_0^t \varphi(t-s) f(s) ds.$$

Formula (7) is exactly the same as (2):

$$\begin{aligned} u(x,t) = &\int_0^t S(x-y,t)\phi(y)dy. \\ &+ \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s) dy\, ds. \end{aligned}$$

The method we have just used to find formula (2) is the operator method.

Proof of (2). All we have to do is verify that the function $u(x,t)$, which is defined by (2), in fact satisfies the PDE and IC (1). Since the solution of (1) is unique, we would then know that $u(x,t)$ is that unique solution. For simplicity, we may as well let $\phi \equiv 0$, since we understand the ϕ term already.

We first verify the PDE. Differentiating (2), assuming $\phi \equiv 0$ and using the rule for differentiating integrals in Section A.3, we have

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s) dy\, ds \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{\partial S}{\partial t}(x-y,t-s) f(y,s) dy\, ds \\ &\quad + \lim_{s\to t}\int_{-\infty}^{\infty} S(x-y,t-s) f(y,s) dy, \end{aligned}$$

Notes

Taking special care due to the singularity of $S(x-y,\mathrm{t}-\mathrm{s})\, at\, t-s=0.$ using the fact that $S(x-y,\mathrm{t}-\mathrm{s})$ satisfies the diffusion equation, we get

$$\frac{\partial u}{\partial t}=\int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2 S}{\partial x^2}(x-y,t-s)\, f(y,s)\,dy\,ds$$
$$+\lim_{\in\to 0}\int_{-\infty}^{\infty} S(x-y,\in)\, f(y,t)\,dy.$$

Pulling the spatial derivative outside the integral and using the initial condition satisfied by S,

we get

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 S}{\partial x^2}\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\, f(y,s)\,dy\,ds+f(x,\mathrm{t})$$
$$=k\frac{\partial^2 u}{\partial x^2}+f(x,t).$$

This identify is exactly the PDE (1). Second, we verify the initial condition. Letting $t\to 0,$ the first term in (2) tends to $\phi(x)$ because of the initial condition of S. the second term is an integral from 0 to 0. Therefore,

$$\lim_{t\to 0} u(x,t)=\phi(x)+\int_0^0 \ldots=\phi(x).$$

This proves that (2) is the unique solution.

Remembering that $S(x,t)$ is the Gaussian distribution , the formula (2) takes the explicit form

$$u(x,t)=\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\, f(y,s)\,dy\,ds$$
$$=\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}}\, e^{-(x-y)^{2/4k(t-s)}}\, f(y,s)\,dy\,ds.$$

in the case that $\phi\equiv 0$.

4.5 SOURCE ON A HALF-LINE

For inhomogeneous diffusion on the half-line we can use the method of reflection.

Now consider the more complicated problem of a boundary source $\mathrm{h}(t)$ on the half-line; that is,

$$\begin{aligned} &\upsilon_t - k\upsilon_{xx} = f(x,t) \qquad \mathit{for}\, 0 < x < \infty, \quad 0 < t < \infty \\ &\upsilon(0,t) = h(t) \\ &\upsilon(x,0) = \phi(x). \end{aligned}$$

We may use the following subtraction device to reduce (9) to a simpler problem. Let $\mathrm{V}(x,t) = \upsilon(x,t) - h(t). \mathit{Then} V(x,t)$ will satisfy

$$\begin{aligned} &V_t - kv_{xx} = f(x,t) - h'(t) \qquad \mathit{for}\, 0 < x < \infty, \quad 0 < t < \infty \\ &V(0,t) = 0 \\ &V(x,0) = \phi(x) - \mathrm{h}(0). \end{aligned}$$

To verify (10), just subtract! This new problem has a homogeneous boundary condition to which we can apply the method of reflection. Once we find V, we recover $\upsilon\, by\, \upsilon(x,t) = V(x,t) + h(t)$.

This simple subtraction device is often used to reduce one linear problem to another.

The domain of independent variables (x,t) in this case is a quarter-plane with specified conditions on both of its half-lines.

If they do not agree at the corner $\left[i.e., if\ \phi(0) \neq h(0)\right]$, then the solution is discontinuous there (but continuous everywhere else). This is physically sensible.

Think for instance, of suddenly at $\mathrm{t} = 0$ sticking a hot iron bar into a cold bath.

For the inhomogeneous Neumann problem on the half-line,

$$\begin{aligned} &w_t - kw_{xx} = f(x,t) \qquad \mathit{for}\, 0 < x < \infty, \quad 0 < t < \infty \\ &w(0,t) = h(t) \\ &w(x,0) = \phi(x). \end{aligned}$$

We would subtract off the function $xh(t)$. That is,

$w(x,t) = w(x,t) - xh(t)$. Differentiation implies that $w_x(0,t) = 0$.

Some of these problems are worked out in the exercise.

EXERCISE

1. Solve the inhomogeneous diffusion equation on the half-line with Dirichlet let boundary condition:

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \qquad (0 < x < \infty,\ 0 < t < \infty) \\ u(0,t) &= 0 \qquad u(x,0) = \phi(x) \end{aligned}$$

2. Using the method of reflection.

Solve the completely inhomogeneous diffusion problem on the half-line

$$\begin{aligned} \upsilon_t - k\upsilon_{xx} &= f(x,t) \qquad \text{for } 0 < x < \infty,\ 0 < t < \infty \\ \upsilon(0,t) &= h(t) \qquad \upsilon(x,0) = \phi(x). \end{aligned}$$

By carrying out the subtraction method begun in the text.

3. Solve the inhomogeneous Neumann diffusion problem on the half-line

$$\begin{aligned} w_t - kw_{xx} &= 0 \qquad \text{for } 0 < x < \infty,\ 0 < t < \infty \\ w_x(0,t) &= h(t) \qquad w(x,0) = \phi(x). \end{aligned}$$

By the subtraction method indicated in the text.

4.6 WAVES WITH A SOURCE

The purpose of this section is to solve

$$u_{tt} - c_2 u_{xx} = f(x,t)$$

on the whole line, together with the usual initial conditions

$$\begin{aligned} u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}$$

Where $f(x,t)$ is a given function? For instance, $f(x,t)$ could be interpreted as external force acting on an infinitely long vibrating string.

Because $L = \partial_t^2 - c^2 \partial_x^2$ is linear operator, the solution will be the sum of three terms. One for ϕ, one for ψ, and one for f. The first two terms are given already in Section 2.1 and we must find the third term. We'll derive the following formula.

Theorem 1: The unique solution of (1),(2) is

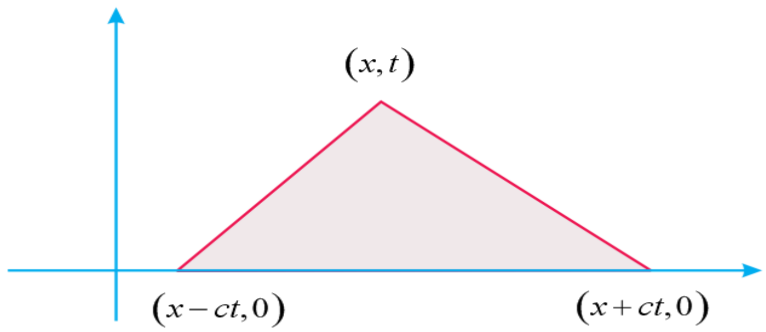
$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-cl}^{x+cl} \psi + \frac{1}{2c}\iint_{\Delta} f$$

Where Δ is the characteristic triangle (see Figure 1).

The double integral in (3) is equal to the iterated integral

$$\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.$$

We will give three different derivations of this formula! But first, let's note what the formula says. It says that the effect of a force f on $u(x,t)$ is obtained



by simply integrating f over the past history of the point (x,t) back to the initial time $t=0$. This is yet another example of the causality principle.

4.7 WELL-POSEDNESS

The well-posedness has three ingredients, as follows. Existence is clear, given that the formula (3) itself is an explicit solution. If ϕ has a continuous second derivative, ψ has a continuous first derivative, and f is continuous, then the formula (3) yields a function u with continuous second partials that satisfies the equation. Uniqueness means that there are no other solutions of (1),(2). This will follow from any one of the derivations given below.

Third, we claim that the problem (1),(2) is stable in the sense of Section 1.5. The means that if the data (ϕ,ψ,f) change a little, then u also changes only a little. To make this precise, we need a way to measure the "nearness" of functions, that is, a metric or norm on function spaces. We will illustrate this concept using the uniform norms:

$$\|w\| = \max_{-\infty<x<\infty} |w(x)|$$

and

$$\|w\|_T = \max_{-\infty<x<\infty, 0\le t\le T} |w(x,t)|.$$

Here T is fixed. Suppose that $u_1(x,t)$ is the solution with data $(\phi_1(x),\psi_1(x),f_1(x,t))$ and $u_2(x,t)$ is the solution with data $(\phi_2(x),\psi_2(x),f_2(x,t))$ (six given functions). We have the same formula (3) satisfied by u_1 and by u_2 except for the different data. We subtract the two formulas. We let $u = u_1 - u_2$. Since the area of Δ equals ct^2, we have from (3) the inequality

$$|u(x,t)| \le \max|\phi| + \frac{1}{2c}.\max|\psi|.2ct + \frac{1}{2c}.\max|f|.ct^2$$
$$= \max|\phi| + t.\max|\psi| + \frac{t^2}{2}.\max|f|.$$

$$\|u_1 - u_2\|_T \le \|\phi_1 - \phi_2\| + T\|\psi_1 - \psi_2\| + \frac{T^2}{2}\|f_1 - f_2\|_T .$$

So if $\|\phi_1-\phi_2\|<\delta, \|\psi_1-\psi_2\|<\delta, and \|f_1-f_2\|_T<\delta, is\, small, \|u_1-u_2\|_T<\delta,\left(1+T+T^2\right)\le\in$

Provided that $\delta\le\in\left(1+T+T^2\right)$. Since $\in$ is arbitrarily small, this argument proves the well-posedness of the problem (1),(2) with respect to the uniform norm.

PROOF OF THEOREM 1

Method of Characteristic Coordinates We introduce the usual characteristic coordinates $\xi=x+ct, \eta=x-ct,$ (see Figure 2).

we have

$$Lu\equiv u_{tt}-c^2u_{xx}=-4c^2u_{\xi\eta}=f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right).$$

We integrate this equation with respect to η, leaving ξ as a constant.

Thus $$u=-\frac{1}{4c^2}\int^{\xi}\int^{\eta} f d\eta d\xi$$

The lower limits this equation here are arbitrary: they correspond to constants of integration. The calculation is much easier to understand if we fix a point p_0 with coordinates x_0, t_0 and

$$\xi_0=x_0+ct_0 \qquad \eta_0=x_0-ct_0.$$

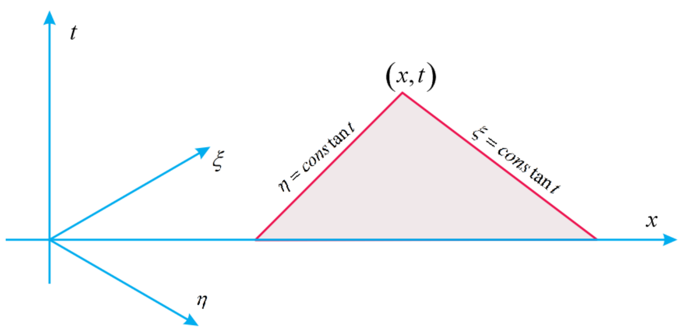


Figure 2

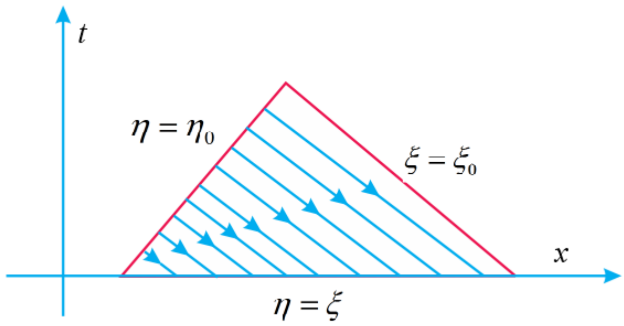


Figure 3

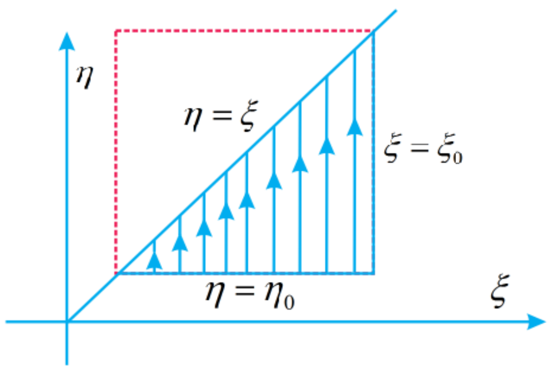
We evaluate (5) at p_0 and make a particular choice of the lower limits. Thus

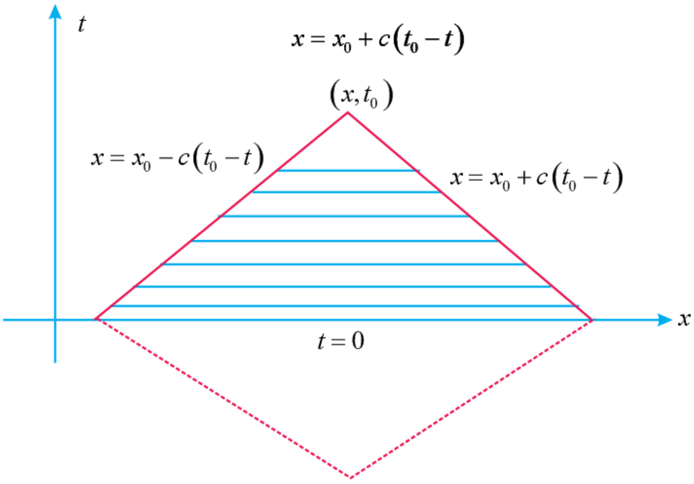
$$u(p_0) = -\frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\xi}^{\eta_0} f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right)d\eta d\xi$$
$$+\frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\xi}^{\eta_0} f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right)d\eta d\xi$$

is particular solution. As Figure 3 indicates, η now represents a variable going along a line segment to the base $\eta = \xi$ of the triangle Δ from the left-hand edge $\eta = \eta_0$, while ξ runs from the left-hand corner to the right-hand edge. Thus we have integrated over the whole triangle Δ .

The iterated integral, however, is not exactly the double integral over Δ because the coordinate axes are not orthogonal. The original axes x *and* y are orthogonal, so we make a change of variables back to x *and* t. This amounts to substituting back

$$x = \frac{\xi+\eta}{2}, \mathrm{t} = \frac{\xi-\eta}{2c}.$$





Our charge of variable is a linear transformation, the Jacobian is just the determinant of its coefficient matrix:

$$J = \left|\det\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial t} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial t} \end{pmatrix}\right| = \left|\det\begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix}\right| = 2c.$$

Thus $d_\eta d\xi = J dx\, dt = 2c\, dx\, dt$. Therefore, the rule for changing variables in a multiple integral (the Jacobian theorem) then gives

$$u(P_0) = \frac{1}{4c^2} \iint_\Delta f(x,t)\, J\, dx\, dt$$

This is precisely Theorem 1. The formula can also be written as the iterated integral in x and t:

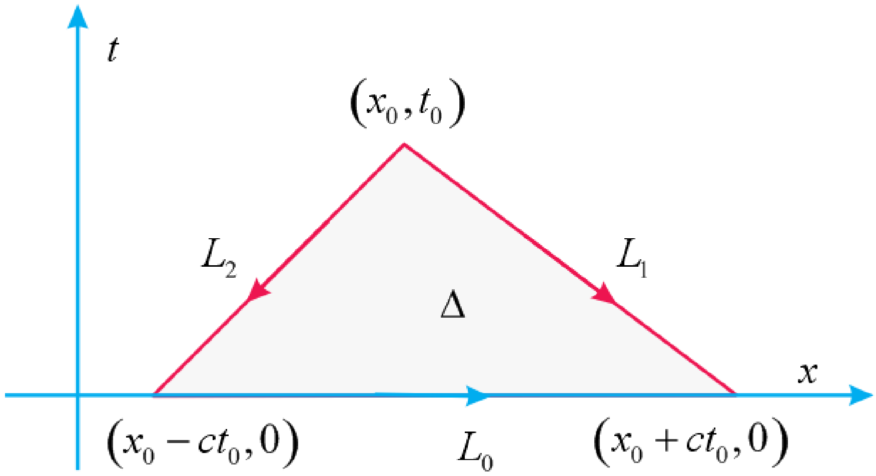
$$u(x_0, t_0) = \frac{1}{2c} \int_0^{t_0} \int_{x_0}^{x_0 + c(t_0 - t)} f(x,t)\, dx\, dt$$

Integrating first over the horizontal line segments in above Figure and the vertically.

A variant of the method of characteristic coordinates is to write 1 as the system of two equations

$$u_t + cu_x = \upsilon \qquad \upsilon_t - c\upsilon_x = f,$$

The first equation being the definition of υ, as in section 2.1 If we first solve the second equation, then υ, is a line integral of f over a characteristic line segment $x + ct =$constant. The first equation then gives $u(x,t)$ by sweeping out these line segments over the characteristic triangle Δ. To carry out this variant is a little tricky, however, and we leave it as an exercise.



Check your progress

2. Prove that $u(x_0,t_0) = \dfrac{1}{2c}\int_0^{t_0}\int_{x_0}^{x_0+c(t_0-t)} f(x,t)\,dx\,dt$

4.8 METHOD USING GREEN'S THEOREM

In this method we integrate f over the past history triangle Δ. Thus

$$\int\int_{\Delta} f\,dx\,dt = \int\int_{\Delta}\left(u_{tt} - c^2 u_{xx}\right)dx\,dt.$$

But Green's theorem says that

$$\int\int_{\Delta}\left(p_x - Q_t\right)dx\,dt = \int\int_{bdy} pdt + Qdx$$

for any functions p *and* Q, where the line integral on the boundary is taken counterclockwise (see Section A.3). Thus we get

$$\int\int_{\Delta} f\,dx\,dt = \int_{L_0+L_2+L_3}\left(-c^2 u_x dt - u_t dx\right).$$

This is the sum of three line integrals over straight line segments (see Figure 6). We evaluate each piece separately. On $L_0, dt = 0\, and\, u_t(x,0) = \psi(x)$, so that

$$\int_{L_0} = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx.$$

On

$L_1, x+ct = x_0 + ct_0, so\, that\, dx + cdt = 0, whence - c^2 u_x dt - u_t dx = cu_x dx + cu_t dt = cdu.$

Thus

$$\int_{L_1} = c\int_{L_1} du = cu(x_0,t_0) - c\phi(x_0+ct_0).$$

In the same way,

$$\int_{L_2} = -c\int_{L_2} du = -c\phi(x_0-ct_0) + cu(x_0,t_0).$$

Adding these three results, we get

$$\int\int_{\Delta} f dx\, dt = 2cu(x_0,t_0) - c\left[\phi(x_0-ct_0) + \phi(x_0-ct_0)\right] - \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx.$$

Thus

$$u(x_0,t_0) = \frac{1}{2c}\int\int_{\Delta} f dx\, dt + \frac{1}{2}\left[\phi(x_0+ct_0) + \phi(x_0-ct_0)\right]$$
$$+\frac{1}{2c}\int_{x_0+ct_0}^{x_0+ct_0} \psi(x)\,dx,$$

Which is the same as before.

Operation Method: This is how we solve the diffusion equation with a source. Let’s try it out on the wave equation. The ODE analog is the equation,

$$\frac{d^2u}{dt^2} + A^2 u(t) = f(t), \quad u(0) = \phi, \;\; \frac{du}{dt}(0) = \psi.$$

We could think of A^2 as a positive constant (or even a positive square matrix.) The solution of (13) is

$$u(t) = S'(t)\phi + S(t)\psi + \int_0^t S(t-s)\, f(s)\,ds,$$

Where

$$S(t) = A^{-1}\sin tA \ and \ S'(t) = \cos tA.$$

The key to understanding formula (14) is that $S(t)\psi$ is the solution of problem (13) in the case that $\phi = 0 \, and \, f = 0.$

Let's return to the PDE

$$u_{tt} - c^2 u_{xx} = f(x,t) \; u(x,0) = \phi(x) \; u_t(x,0) = \psi(x).$$

The basic operator ought to be given by the ψ term. That is,

$$\varphi(t)\psi = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy = \upsilon(x,t),$$

Where $\upsilon(x,t)$ solves $\upsilon_{tt} - c^2\upsilon_{xx} = 0, \upsilon(x,0) = 0, \upsilon_t(x,0) = \psi(x). \varphi(t)$ is the source operator. By (14) we would expect the ϕ term to be $(\partial/\partial t)\varphi(t)\phi.$ In fact,

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(t)\phi &= \frac{\partial}{\partial t}\frac{1}{2c}\int_{x-ct}^{x+ct}\phi(y)\,dy \\ &= \frac{1}{2c}\left[cgf(x+ct) - (-c)\phi(x-ct)\right], \end{aligned}$$

In agreement with our old formula (2.1.8)! So we must be on the right track.

Let's now take the f term; that is, $\phi = \psi = 0.$ By analogy with the last term in (14), the solution ought to be

$$u(t) = \int_0^t \varphi(t-s)\,f(s)\,ds.$$

That is, using (17),

$$u(x,t) = \int_0^t \left[\frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\right]ds = \frac{1}{2c}\int \int_\Delta f \, dx\,dt.$$

This is once again the same result!

The moral of the operator method is that if you can solve the homogeneous equation, you can also solve the inhomogeneous equation. This is sometimes known as Duhamel's principle.

SOURCE ON A HALF-LINE

The solution of the general inhomogeneous problem on a half-line

$$
\begin{aligned}
&DE: \upsilon_{tt} - c^2\upsilon_{xx} = f(x,t) \, in \; 0 < x < \infty \\
&IC: \upsilon(x,0) = \phi(x) \; \upsilon_t(x,0) = \psi(x) \\
&BC: \upsilon(0,t) = h(t)
\end{aligned}
$$

is the sum of four terms, one for each data function $\phi, \psi, f, and\ h. For\ x > ct > 0,$ the solution has precisely the same form as in (3), with the backward triangle Δ as the domain of dependence. For $0 < x < ct,$ however, it is given by

$$\upsilon(x,t) = \phi\, term + \psi\, term + h\left(t - \frac{x}{c}\right) + \frac{1}{2c}\int\int_D f$$

Where $t - x/c$ is the reflection point and D is the shaded region in Figure 3.2.2. The only caveat is that the given conditions had better coincide at the origin. That is, we require that $\phi(0) = h(0)\, and\, \psi(0) = h'(0).$ If this were not assumed, there would be a singularity on the characteristic line emanating from the corner.

Let's derive the boundary term $h\left(t - x/c\right) for\ x < ct.$ To accomplish this, it is convenient to assume that $\phi = \psi = f = 0.$ We shall derive the solution from scratch using the fact that $\upsilon(x,t)$ must take the form $\upsilon(x,t) = j(x+ct) + g(x-ct).$ From the initial conditions $(\phi = \psi = 0),$ we find that $j(s) = g(s) = 0\ for\ s > 0.$ From the boundary condition we have $h(t) = \upsilon(0,t) = g(-ct)\ for\ t > 0.$ Thus $g(s) = h\left(-s/c\right) for\ s < 0.$

Therefore, if $x < ct, t > 0, we\, have\, \upsilon(x,t) = 0 + h\left(-[x-ct]/c\right) = h\left(t - x/c\right).$

FINITE INTERVAL

Notes

For a finite interval $(0,l)$ with inhomogeneous boundary conditions $\upsilon(0,t)=h(t), \upsilon(l,\mathrm{t})=k(t)$, we get the whole series of terms

$$\upsilon(l,\mathrm{t})=h\left(t-\frac{x}{c}\right)-h\left(t+\frac{x-2l}{c}\right)+h\left(t-\frac{x+2l}{c}\right)+\ldots$$
$$+k\left(t+\frac{x-1}{c}\right)-k\left(t-\frac{x+l}{c}\right)+k\left(t+\frac{x-3l}{c}\right)+\ldots$$

EXERCISE:

1. Solve $u_{tt}=c^2u_{xx}+xt,\ u(x,0)=0, u_t(x,0)=0.$
2. Solve $u_{tt}=c^2u_{xx}+e^{ax},\ u(x,0)=0,\ u_t(x,0)=0.$
3. Solve $u_{tt}=c^2u_{xx}+Cosx,\ u(x,0)=\sin x,\ u_t(x,0)=1+x.$
4. Show that the solution of the inhomogeneous wave equation

   $$u_{tt}=c^2u_{xx}+f,\ and\ \ u(x,0)\equiv\phi(x), u_t(x,0)=\psi(x).$$

   Is the sum of three terms, one each for $f, \phi, and\, \psi$.

5. Let $f(x,t)$ be any function and let $u(x,t)=\left(\frac{1}{2c}\right)\int\int_\Delta f, where\,\Delta$ is the triangle of dependence. Verify directly by differentiation that

   $$u_{tt}=c^2u_{xx}+f\ and\ u(x,0)\equiv u_t(x,0)\equiv 0$$

   (Hint: Begin by writing the formula as the iterated integral

   $$u(x,t)=\frac{1}{2c}\int\int_{x-ct+cs}^{x+ct-cs}f(y,s)\,dy\,ds$$

6. Derive the formula for the inhomogeneous wave equation in yet another way.

   Write it as the system

   $$u_t+cu_x=\upsilon,\quad \upsilon_t-c\upsilon_x=f.$$

7. Solve the first equation for u in terms of υ as

   $$u(x,t)=\int_0^t\upsilon(x-ct+cs,s)\,ds.$$

   Similarly, solve the second equation for υ in terms of f .

8. Let A be a positive-definite $n\times n$ matrix. Let

$$S(t)=\sum_{m=0}^{\infty}\frac{(-1)^m A^{2m}t^{2m+1}}{(2m+1)!}.$$

9. Show that this series of matrices converges uniformly for bounded t and its sum $S(t)$ solves the problem $S''(t)+A^2S(t)=0, S(0)=0, S'(0)=I, where I$ is the identify matrix. Therefore, it makes sense to denote $S(t)\, as\, A^{-1}\sin tA$ and to denote its derivative $S'(t)\, as \cos(tA)$.

10. Show that the source operator for the wave equation solves the problem

$$\varphi_{tt}-c^2\varphi_{xx}=0,\ \varphi(0)=0,\ \varphi_t(0)=I,$$

Where I is the identify operator.

11. Use any method to show that $u={}^{1}\!/\!_{(2c)}\int\int_D f$ solves the inhomogeneous wave equation on the half-line with zero initial boundary data, where D is the domain of dependence for the half-line.

12. Show by direct substitution that $u(x,t)=h\left(t-{}^{x}\!/\!_{c}\right)\ for\ x<ct\ and\ u(x,t)=0\ for\ x\geq ct$ solves the wave equation on the half-line $(0,\infty)$ with zero initial data and boundary condition $u(0,t)=h(t).$

13. Derive the solution of the fully inhomogeneous wave equation on the half-line

$$\begin{aligned}\upsilon_{tt}-c^2\upsilon_{xx} &= f(x,t)\quad in\, 0<x<\infty\\ \upsilon(x,0)=\phi(x), &\quad \upsilon_t(x,0)=\psi(x)\\ &=\upsilon(0,t)=h(t),\end{aligned}$$

by means of the method using Green's theorem. (Hint: Integrate over the domain of dependence.)

14. Solve

Notes

$$u_{tt} = c^2 u_{xx} \; for \quad 0 < x < \infty,$$
$$u(0,t) = t^2, \quad u(x,0) = x, u_t(x,0) = 0.$$

the homogeneous wave equation on the half-line $(0,\infty)$ with zero initial data and with the Neumann boundary condition $u_x(0,t) = k(t)$. Use any method you wish.

15. Derive the solution of the wave equation in a finite interval with inhomogeneous boundary conditions $\upsilon(0,t) = h(t), \upsilon(l,t) = k(t), and\ with\ \phi = \psi = f = 0.$

DIFFUSION REVISTED

In this section we make a careful mathematical analysis of the solution of the diffusion equation that we found in previous chapter. (On the other hand, the formula for the solution of the wave equation is so much simpler that it doesn't require a special justification.)

The solution formula for the diffusion equation is an example of a convolution, the convolution of ϕ with S (at a fixed t). It is

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy = \int_{-\infty}^{\infty} S(z,t)\phi(x-z)\,dz,$$

Where $S(z,t) = 1 \big/ \sqrt{4\pi k t e^{-z^2/4kt}}.$ If we introduce the variable $p = z \big/ \sqrt{kt},$ it takes the equivalent form

$$u(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^{2/4}} \phi\left(x - p\sqrt{kt}\right) dp.$$

Now we are prepared to state a precise theorem.

Theorem1. Let $\phi(x)$ be a bounded continuous function for $-\infty < x < \infty$. Then the formula (2) defines an infinitely differentiable function $u(x,t) \; for -\infty < x < \infty,$ which satisfies the equation $u_t = ku_{xx} \; and \lim_{t \to 0} u(x,t) = \phi(x)$ for each x .

Proof: The integral converges easily because

$$\left|u(x,t)\right| \le \frac{1}{\sqrt{4\pi}}\left(\max|\phi|\right)\int_{-\infty}^{\infty} e^{-p^2/4}dp = \max|\phi|.$$

(This inequality is related to the maximum principle.) Thus the integral converges uniformly and absolutely. Let us show that $\partial u/\partial x$ exists. It equals $\int\left(\partial S/\partial x\right)(x-y,t)\phi(y)\,dy$ provided that this new integral also converges absolutely. Now

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\partial S}{\partial x}(x-y,t)\phi(y)\,dy &= -\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}\frac{x-y}{2kt}e^{-(x-y)^2/4kt}\phi(y)\,dy\\ &= \frac{c}{\sqrt{t}}\int_{-\infty}^{\infty} pe^{-p^2/4}\phi\left(x-p\sqrt{kt}\right)\\ &\le \frac{c}{\sqrt{t}}\left(\max|\phi|\right)\int_{-\infty}^{\infty}|p|e^{-p^2/4}dp,\end{aligned}$$

Where c is a constant. The last integral is finite. So this integral also converges uniformly and absolutely. Therefore, $u_x = \partial u/\partial x$ exists and is given by this formula. All derivatives of all orders $(u_t, u_{xt}, u_{xx}, u_{tt}, \ldots)$ work the same way because each differentiation brings down a power of p so that we end up with convergent integrals like $\int p^n e^{-p^{2/4}}dp. so\, u(x,t)$ is differentiable to all orders. Since $S(x,t)$ satisfies the diffusion equation for $t>0,$ so does $u(x,t)$.

It remains to prove the initial condition. It has to be understood in a limiting sense because the formula itself has because the formula itself has meaning only for $t>0.$ Because the integral of $S\, is\, 1,$ we have

$$\begin{aligned}u(x,t)-\phi(x) &= \int_{-\infty}^{\infty} S(x-y,t)\left[\phi(y)-\phi(x)\right]dy\\ &\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty}e^{-p^2/4}\left[\phi\left(x-p\sqrt{kt}\right)-\phi(x)\right]dp.\end{aligned}$$

For fixed x we must show that this tends to zero as $t\to 0.$ The idea is that for $p\sqrt{t}$ small, the continuity of ϕ make the integral small; while for $p\sqrt{t}$ not small, p is large and the exponential factor is small.

Notes

To carry out this idea, let $\in > 0. Let\, \delta > 0$ be so small that

$$\max_{|y-x|\le\delta} \left|\phi(y)-\phi(x)\right| < \frac{\in}{2}.$$

This can be done because ϕ is continuous at x. We break up the integral into the part where $|p| < \delta/\sqrt{kt}$ and the part where $|p| \ge \delta/\sqrt{kt}$. The first part is

$$\left|\int_{|p|<\delta/\sqrt{kt}}\right| \le \left(\frac{1}{\sqrt{4\pi}}\int e^{-p^2/4}\right).\max_{|y-x|\le\delta}\left|\phi(y)-\phi(x)\right|$$

$$< 1.\frac{\in}{2} = \frac{\in}{2}.$$

The second part is

$$\left|\int_{|p|\ge\delta/\sqrt{kt}}\right| \le \frac{1}{\sqrt{4\pi}}.2\left(\max|\phi|\right).\int_{|p|\ge\delta/\sqrt{kt}} e^{-p^2/4}dp < \frac{\in}{2}$$

By choosing t sufficiently small, since the integral $\int_{\infty}^{\infty} e^{-p^2/4}dp$ converges and δ is fixed. (That is, the "tails") $\int_{|p|\ge N} e^{-p^2/4}dp$ are as we wish if $N = \dfrac{\delta}{\sqrt{kt}}$ is large enough.) Therefore,

$$\left|u(x,t)-\phi(x)\right| < \frac{1}{2}\in + \frac{1}{2}\in = \in$$

Provided that t is small enough. This means exactly that $u(x,t) \to \phi(x)$ as $t \to 0.$

Corollary: The solution has all derivatives of all orders for $t > 0.$ even if ϕ is not differentiable. We can say therefore that all solutions become smooth as soon as diffusion takes effect. There are no singularities, in sharp contrast to the wave equation.

Proof : We use formula $u(x,t)\int_{-\infty}^{\infty} s(x-y,t)\phi(y)\,dy$

Together with the rule for differentiation under an integral sign, Theorem 2 in section A.3.

Piecewise Continuous Initial Data: Notice that the continuity of $\phi(x)$ was used in only one part of the proof. With an appropriate change we can allow $\phi(x)$ to have a jump discontinuity. [Consider, for instance, the initial data for $Q(x,t)$.]

A function $\phi(x)$ is said to have a jump at x_0 if both the limit of $\phi(x)$ as $x \to x_0$ from the right exists [denoted $\phi(x_0 +)$] exists but these two limits are not equal. A function is called piecewise continuous if in each finite interval it has only a finite number of jumps and it is continuous at all other points. This concept is discussed in more detail in Section 5.2.

Theorem 2 : Let $\phi(x)$ be a bounded function that is piecewise continuous. Then (1) is an infinitely differentiable solution for $t > 0$ and

$$\lim_{x\to 0} u(x,t) = \frac{1}{2}\left[\phi(x+) + \phi(x-)\right]$$

For all x. At every point of continuity this limit equals $\phi(x)$.

Proof : The idea is the same as before. The only difference is to split the integrals into $p > 0$ *and* $p < 0$. We need to show that

$$\frac{1}{\sqrt{4\pi}} \int_0^{\pm\infty} e^{-p^2/4} \phi\left(x + \sqrt{kt}\, p\right) dp \to \pm \frac{1}{2} \phi(x \pm).$$

The details are left as an exercise.

EXERCISE:

1. Prove that if ϕ is any piecewise continuous function, then

$$\frac{1}{\sqrt{4\pi}} \int_0^{\pm\infty} e^{-p^2/4} \phi\left(x + \sqrt{kt}\, p\right) dp \to \pm \frac{1}{2} \phi(x \pm) \text{as } t \to 0.$$

2. Use Exercise 1 to prove Theorem 2.

Check your progress

3. Explain the method using Green's theorem.

   ----

4. Prove the theorem Let $\phi(x)$ be a bounded continuous

function for $-\infty < x < \infty$. Then the formula (2) defines an infinitely differentiable function $u(x,t)$ *for* $-\infty < x < \infty$, which satisfies the equation $u_t = ku_{xx}$ *and* $\lim_{t \to 0} u(x,t) = \phi(x)$ for each x .

4.9 LET US SUM UP

In this unit we have discussed about Diffusion on the half-line, Reflection of waves, Diffusion with a source, Source on a half-line, Waves with a source, Well-posedness, Method using

Green's theorem and The reflection method. The odd extension of both of the initial functions to the whole line. Homogeneous diffusion equation on the whole line. Use the method of reflection for inhomogeneous diffusion on the half-line. The solution formula for the diffusion equation. Piecewise continuous initial data.

4.10 KEY WORDS

1. Single point of reflection at one end of a semi-infinite line.
2. The odd extension of both of the initial functions to the whole line.
3. In homogeneous diffusion equation on the whole line
4. Our charge of variable is a linear transformation; the Jacobian is just the determinant of its coefficient matrix:
5. Method using Green's Theorem In this method we integrate f over the past history triangle Δ. Thus

$$\int\int_{\Delta} f\,dx\,dt = \int\int_{\Delta}\left(u_{tt} - c^2 u_{xx}\right)dx\,dt.$$

6. Source on a half line
7. Waves with a source

4.11 QUESTIONS FOR REVIEW

1. Discuss diffusion on the half-line
2. Discuss diffusion with a source
3. Discuss reflection of waves
4. Discuss source on a half line
5. Discuss method using Green's Theorem

4.12 SUGGESTED READINGS AND REFERENCES

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10. Partial Differential Equations,*Erich Mieremann*
11. Partial Differential Equations,-*Victor Ivrii*

4.13 ANSWERS TO CHECK YOUR PROGRESS

1. See section 4.4
2. See section 4.8
3. See section 4.9
4. See section 4.9

UNIT-5 BOUNDARY PROBLEMS

STRUTURE

5.0 OBJECTIVE

After studying this unit we will learn about
separation of variables, the Dirichlet condition,
The Neumann condition,
The Robin condition, Positive eigen values, Zero eigen values.

5.1 INTRODUCTION

In this chapter we finally come to the physically realistic case of a finite interval $0 < x < l.$ The methods we introduce will frequently be used in the rest of this book.

5.2 SEPARATION OF VARIABLES, THE DIRICHLET CONDITION

We first consider the homogeneous Dirichlet conditions for the wave equation:

$$u_{tt} = c^2 u_{xx} \quad for\, 0 < x < l \;---(1)$$
$$u(0,t) = 0 = u(l,t) ---(2)$$

With some initial conditions

$$u(0,t) = \phi(x) u_t(\mathrm{x},0) = \psi(x).\text{--------}(3)$$

The method we shall use consists is building up the general solution as a linear combination of special ones that are easy to find.

A separated solution is a solution of (1) and (2) of the form

$$u(x,t) = X(x)T(t).\text{----------}(4)$$

It is important to distinguish between the independent variable written as a lower case letter and the function written as a capital letter.

Our first goal is to look for as many separated solutions as possible.

Plugging the form (4) into the wave equation (1), we get

$$X(x)T''(t)c^2X''(x)t(t)$$

Or, dividing by $-c^2XT$,

$$-\frac{T''}{c^2T} = -\frac{X''}{X} = \lambda.$$

This defines a quantity λ, which must be a constant, (Proof: $\partial\lambda/\partial x = 0\, and\, \partial\lambda/\partial t = 0, so\, \lambda$ is constant.

Alternatively, we can argue that λ doesn't depend on x because of the first expression and doesn't depend on t because of the second expression, so that it doesn't depend on any variable.

We will show at the end of this section that $\lambda > 0.$

So let $\lambda = \beta^2, where\, \beta > 0.$ (This the equation above are a pair of separate (!) ordinary differential equations for $X(x)\, and\, T(t)$:

$$X'' + \beta^2 X = 0\, and\, T'' + c^2\beta^2 T = 0.\text{--------}(5)$$

These ODEs are easy to solve. The solutions have the form

Notes

$$X(x) = C\cos\beta x + D\sin\beta x --- (6)$$
$$T(t) A\cos\beta ct + B\sin\beta ct, --- (7)$$

Where A, B, C, and D are constants.

The second step is to impose the boundary conditions (2) on the separated solution. They simply require that $X(0) = 0 = X(1)$.

$$0 = X(0) = C \ and \ 0 = X(l) = \mathrm{D}\sin\beta l.$$

Surely we are not interested in the obvious solution $C = D = 0.$ So we must have $\beta l = n\pi,$ a root of the sine function. That is,

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \ X_n(x) = \sin\frac{n\pi x}{l} \ (n = 1, 2, 3, ...) \ldots\ldots.(8)$$

are distinct solutions. Each sine function may be multiplied by an arbitrary constant.

Therefore, there are an infinite (!) number of separated solution of (1) and (2), one for each n. They are

$$u_n(x,t) = \left(A_n \cos\frac{n\pi}{l} + B_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

(n=1,2,3,…), where $A_n \ and \ B_n$ are arbitrary constants. The sum of solutions is again a solution, so any finite sum.

$$u(x,t) = \sum_n \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \text{-------}(9)$$

is also a solution of (1) and (2).

Formula (9) solves (3) as well as (1) and (2), provided that

$$\phi(x) = \sum_n A_n \sin\frac{n\pi x}{l} --- (10)$$

and

$$\psi(x) = \sum_n \frac{n\pi c}{l} B_n \sin\frac{n\pi x}{l}. --- (11)$$

Thus for any initial data of this form, the problem (1), (2), and (3) has a simple explicit solution.

But such data (10) and (11) clearly are very special. So let's try (following Fourier in 1827) to take infinite sums. Then we ask what kind of data pairs $\phi(x), \psi(x)$ can be expanded as in (10), (11) for some choice of coefficients A_n, β_n?

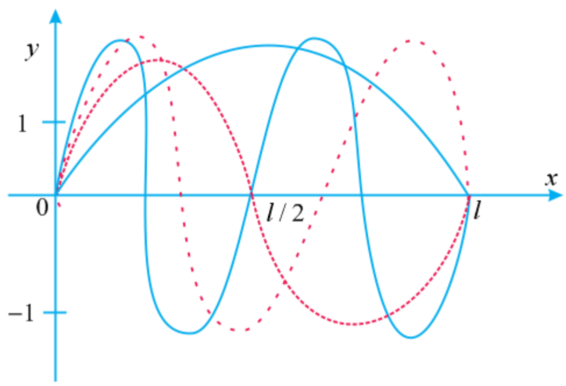
This question was the source of great disputes for half a century around 1800, but the final result of the disputes was very simple: Practically any (!) function $\phi(x)$ on the interval $(0, l)$ can be expanded in an infinite series (10).

It will have to involve technical questions of convergence and differentiability of infinite series like (9). The series in (10) is called a Fourier since series on $(0, l)$.

But for the time being let's not worry about these mathematical points. Let's just forge ahead to see what their implications are.

First of all, (11) is the same kind of series for $\psi(x)$ as (10) is for $\phi(x)$. What we've shown is simply that if (10), (11) are true, then the infinite series (9) ought to be the solution of the whole problem (1), (2), (3).

A sketch of the first few functions $\sin\left(\pi x/l\right), \sin\left(2\pi x/l\right), \ldots$ is shown in Figure 1. The functions $\cos\left(n\pi ct/l\right)$ *and* $\sin\left(n\pi x/l\right), \ldots$ which describe the behavior in time have a similar form. The coefficients of t inside the sines and cosines, namely $n\pi ct/l$, are called the frequencies. (In some texts, the frequency is defined as $nc/2l$.



Notes

Figure 1

If we return to the violin string that originally led us to the problem (1), (2), (3), we find that the frequencies are

$$\frac{n\pi\sqrt{T}}{t\sqrt{p}} \; for\; n = 1, 2, 3, \ldots \text{-------}(12)$$

The "fundamental" note of the string is the smallest of these, $\pi\sqrt{T}/\left(t\sqrt{p}\right)$. The "overtones" are exactly the double, the triple, and so on, of the fundamental! The discovery by Euler in 1749 that the musical notes have such a simple mathematical description created a sensation. It took over half a century to resolve the ensuing controversy over the relationship between the infinite series (9) and d' Alembert's solution in Section 2.1.

The analogous problem for diffusion is

$$DE: u_t = ku_{xx} \; \left(0 < l, o < t < \infty\right) --- (13)$$
$$BC: u\left(0,t\right) = u\left(l,t\right) = 0 --- (14)$$
$$IC: u\left(x,0\right) = \phi\left(x\right) --- (15)$$

To solve it, we separate the variables $u = T\left(t\right)X\left(x\right)$ as before. This time we get

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda = cons\tan t.$$

Therefore, $T\left(t\right)$ satisfies the equation $T' = -\lambda kT$, whose solution is $T\left(t\right) = A^{e^{-\lambda kt}}$. Furthermore,

$$-X'' = \lambda X \;\; in\, 0 < X < l \;\; with \; x\left(0\right) = X\left(l\right) = 0.\text{-----}(16)$$

This is precisely the same problem for $X\left(x\right)$ as before and so has the same solutions. Because of the form of $T\left(t\right)$,

$$u\left(x,t\right) = \sum_{n=1}^{\infty} A_n e^{-\left(n\pi/l\right)^2 kt} \sin\frac{n\pi x}{l} \;.\text{-----}(17)$$

is the solution of (13) - (15) provided that

$$\phi(x)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}. \text{------}(18)$$

Once again, our solution is expressible for each t as a Fourier sine series in x provided that the initial data are.

For example, consider the diffusion of a substance in a tube of length l. Each end of the tube opens up into a very large empty vessel. So the concentration $u(x,t)$ at each end is essentially zero. Given an initial concentration $\phi(x)$ in the tube, the concentration at all later times is given by formula (17). Notice that as $t\to\infty$, each term in (17) goes zero. Thus the substance gradually empties out into the two vessels and less and less remains in the tube.

The numbers $\lambda_n=\left(n\pi/l\right)^2$ are called eigenvalues and the functions $X_n(x)=\sin\left(n\pi/l\right)$ are called eigen functions. The reason for this terminology is as follows. They satisfy the conditions

$$-\frac{d^2}{dx^2}X=\lambda X, \quad X(0)=X(l)=0. \text{------}(19)$$

The is an ODE with conditions at two points. Let A denote the operator $-d^2/dx^2$, which acts on the functions that satisfy the Dirichlet boundary conditions. The differential equation has the form $AX=\lambda X$. An eigen function is a solution $X\neq 0$ of this equation and an eigenvalue is a number λ for which there exists a solution $X\neq 0$.

This situation is analogous to the more familiar case of an $N\times N$ matrix A. A vector X that satisfies $AX=\lambda X \; with \; X\neq 0$ is called an eigenvector and λ is called an eigenvalue. For an $N\times N$ matrix there are at most N eigenvalues. But for the differential operator that we are interested in, there are an infinite number of eighn values $\pi^2/l^2, 4\pi^2/l^2, 9\pi^2/l^2, \ldots$. Thus you might say that we are dealing with infinite-dimensional linear algebra!

Notes

In physics and engineering the engenfunctions are called normal modes because they are the natural shapes of solutions that persist for all time.

Why are all the eigenvalues of this problem positive? We assumed this in the discussion above, but let's prove it. First, could $\lambda = 0$ be an eigenvalue? This would mean that $X'' = 0,$ so that $X(x) = C + Dx.$ But $X(0) = X(l) = 0$ implies that $C = D = 0,$ so that $X(x) \equiv 0.$ Therefore, zero is not an eigenvalue.

Next, could there be negative eigenvalues? If $\lambda < 0,$ let's write it as $\lambda = -\gamma^2 . Then\, X'' = \gamma^2 X,$ so that $X(x) = C \cosh \gamma x + D \sinh \gamma l.$ Hence $D = 0$ since $\sinh \gamma l \neq 0.$

$$X(x) = Ce^{\gamma x} + De^{-\gamma x},$$

Where we are using the complex exponential function.

The boundary conditions yield $0 = X(0) = C + D\, and\, 0 = Ce^{\gamma l}.$ Therefore $Ce^{\gamma l} = 1.$ By a well-known property of the complex exponential function, this implies that $\mathrm{Re}(\gamma) = 0\, and\, 2lIM(\gamma) = 2\pi n$ for some integer n. Hence $\gamma = n\pi i / l\; and\; \lambda = -\gamma^2 = n^2\pi^2 / l^2,$ which is real and positive . Thus the only eigenvalues λ of our problem (16) are positive numbers; in fact, they are $\left(\pi / l\right)^2, \left(2\pi / l\right)^2, \ldots.$

EXERCISE:

1. (a) Use the Fourier expansion to explain why the note produced by a violin string rises sharply by one octave when the string is clamped exactly at its midpoint.

   (b) Explain why the note rises when the string is tightened.

2. Consider a metal rod $(0 < x < l),$ insulated along its sides but not at its ends, which is initially at temperature=0. Suddenly both ends are plunged into a bath of temperature=0. Wrote the

formula for the temperature $u(x,t)$ at later times. In this problem, assume the infinite series expansion

$$t = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \ldots\right)$$

3. A quantum-mechanical particle on the line with an infinite potential outside the interval $(0,l)$ ("particle in a box") is given by Schtrodinger's equation $u_t = iu_{xx}$ on $(0,l)$ with Dirichlet conditions at the ends. Separate the variable and use (8) to find its representation as a series.

4. Consider eaves in a resistant medium that satisfy the problem

$$u_{tt} = c^2 u_{xx} - \gamma u_t \quad \mathit{for}\ 0 < x < l$$
$$u = 0 \quad \mathit{at\ both\ ends}$$
$$u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x),$$

Where γ is a constant, $0 < \gamma < {}^{2\pi c}\!/_{l}$. Write down the series expansion of the solution.

5. Do the same for ${}^{2\pi c}\!/_{t} < \gamma < {}^{4\pi c}\!/_{l}$.

6. Separate the variables for the equation $-1 \le x \le l.$ with the boundary conditions $u(0,t) = u(\pi,t) = 0.$ Show that there are an infinite number of solutions that satisfy the initial condition $u(\mathrm{x},t) = 0$. So uniqueness is false for this equation.

Check your progress

Explain about the Dirichlet conditions for the wave equation

5.3 THE NEUMANN CONDITION

Notes

The same method works for both the Neumann and Robin boundary conditions (BCs). In the former case, (4.1.2) is replaced by $u_x(0,t)=u_x(l,t)=0.$ then the eigen functions are the solutions $X(x)$ of $-X''=\lambda X, \quad X'(0)=X'(l)=0,$--------(1)

Other than the trivial solution $X(x)\equiv 0.$

As before, let's first search for the positive eigenvalues $\lambda=\beta^2>0. As\, in(4.1.6), x(x)=C\cos\beta x+D\sin\beta x,$ so that

$$X'(x)=-C\beta\sin\beta x+D\beta\cos\beta x.$$

The boundary conditions (1) mean first that $0=X'(0)=D\beta,$ so that $D=0,$ and second that

$$0=X'(l)=-C\beta\sin\beta l.$$

Since we don't want $C=0,$ we must have $\sin\beta l=0.$ Thus $\beta=\pi/l, 2\pi/l, 3\pi/l, ...$ Therefore, we have the

$$Eligenvalues:\left(\frac{\pi}{l}\right)^2,\left(\frac{2\pi}{l}\right)^2,---(2)$$

$$Eigenfunctions: X_n(x)=\cos\frac{n\pi x}{l}(n=1,2,...)---(3)$$

Next let's check whether zero is an eigenvalue. Set $\lambda=0$ in the ODE (1).Then $X''=0$. So that $X(x)=C+Dx\, and\, X'(x)\equiv D.$ The Neumann boundary conditions are both satisfied if $D=0.C$ can be any number. Therefore, $\lambda=0$ is an eigenvalue, and any constant function is its eigenfunction.

If $\lambda<0$ or if λ is complex (nonreal), it can be shown directly, as in the Dirichlet case, that there is no eigenfunction. (Another proof will be given in Section 5.3.) Therefore, the list of all the eigenvalues is

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2 \; for\, n=0,1,2,3,...\text{-------}(4)$$

Note than $n=0$ is included among them!

So, for instance, the diffusion equation with the Neumann BCs has the solution

$$u(x,t)=\frac{1}{2}A_0+\sum_{n=l}^{\infty}A_n e^{-\left(n\pi/l\right)^2 kt}\cos\frac{n\pi x}{l}. \text{ ------(5)}$$

This solution requires the initial data to have the "Fourier cosine expansion"

$$\phi(x)=\frac{1}{2}A_0+\sum_{n=l}^{\infty}A_n\cos\frac{n\pi x}{l}. \text{--------(6)}$$

All the coefficients $A_0, A_1, A_2, ...$ are just constants. The first term in (5) and (6), which comes from the eigenvalue $\lambda=0,$ is written separately in the form $\frac{1}{2}A_0$ just for later convenience. (The reader is asked to bear with this ridiculous factor $\frac{1}{2}$, when its convenience will become apparent.)

What is the behavior of $u(x,t)\, as\, t\to\infty$? Since all but the first term in (5) contains an exponentially decaying factor, the solution decays quite fast to the first term $\frac{1}{2}A_0,$ which is just a constant. Since these boundary conditions correspond to insulation at both ends, this agrees perfectly with our intuition of Section 2.5 that the solution "spreads out." This is the eventual behavior if we wait long enough. (To actually prove that the limit as $t\to\infty$ is given term by term in (5) requires the use of one of the convergence theorems in Section A.2. We omit this verification here.)

Consider now the wave equation with the Neumann BCs. The eigenvalue $\lambda=0$ then leads to $x(x)=$ constant and to the differential equation $T''(t)=\lambda c^2 T(t)=0,$ which has the solution $T(t)=A+Bt.$ Therefore, the wave equation with Neumann BCs has the solutions

Notes

$$u(x,t)=\frac{1}{2}A_0+\frac{1}{2}B_0t$$
$$+\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{l}+B_n\sin\frac{n\pi ct}{l}\right)\cos\frac{n\pi x}{l}. \text{---------(7)}$$

(Again, the factor $\frac{1}{2}$ will be justified later.) Then the initial data must satisfy

$$\phi(x)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}A_n\cos\frac{n\pi x}{l}---(8)$$

and

$$\psi(x)=\frac{1}{2}B_0+\sum_{n=1}^{\infty}\frac{n\pi c}{l}B_n\cos\frac{n\pi x}{l}---(9)$$

Equation (9) comes from first differentiating (7) with respect to t and then setting $t=0.$

A "mixed: boundary condition would be Dirichlet at one end and Neumann at the other. For instance, in case the BCs are $u(0,t)=u_x(l,t)=0,$ the eigenvalue problem is

$$-X''=\lambda x\ X(0)=x'(l)=0.\text{---------(10)}$$

The eigenvalues then turn out to be $\left(n+\frac{1}{2}\right)^2\pi^2/l^2$ and the eigen function $\sin\left[\left(n+\frac{1}{2}\right)^2\pi x/l\right]$ $for\ n=0,1,2,...$ (see Exercises 1 and 2).

For a discussion of boundary conditions in the context of musical instruments, see $[HJ]$.

For another example, consider the Schrodiner equation $u_t=iu_{xx}$ in $(0,l)$ with the Neumann $BCs\,u_x(0,t)=u_x(l,t)=0$ and initial condition $u(x,0)=\phi(x).$

$$\frac{T'}{iT}=\frac{X''}{x}=-\lambda=cons\tan t,$$

So that $T(t)=e^{-i\lambda t}$ and $X(x)$ satisfies exactly the same problem (1) as before. Therefore, the solution is

$$u(x,t)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}A_n e^{-i\left(n\pi/l\right)^2 t}\cos\frac{n\pi x}{l}.$$

The initial condition requires the cosine expansion (6).

EXERCISE

1. Solve the diffusion problem $u_t = ku_{xx}\, in\, 0, x<l,$ with the mixed boundary conditions $u(0,t)=u_x(l,t)=0.$

2. Consider the equation $u_{tt}=c^2u_{xx}\ for\, 0<x<l,$ with the boundary conditions $u_x(0,t)=0, u_x(l,t)=0$ (Neumann at the left, Dirichlet at the right).

(a)Show that the eigen functions are $\cos\left[\left(n+\frac{1}{2}\right)\pi x/l\right].$

(b) Write the series expansion for a solution $\mathrm{u}(x,t).$

3. Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-1\le x\le l.$ Then the concentration of the diffusing substance satisfies

$$u_t = ku_{xx}\quad for\ -l\le x\le l$$
$$u(-l,t)=u(l,t)\ \ and\ \ u_x(-l,t)=u_x(l,t).$$

These are called periodic boundary conditions.

(a)Show that the eigenvalues are $\lambda=\left(n\pi/l\right)^2\ for\, n=0,1,2,3,...$

(b) Show that the concentration is

$$u(x,t)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{l}+B_n\sin\frac{n\pi x}{l}\right)e^{-n^2\pi^2kt/l^2}.$$

5.4 THE ROBIN CONDITION

We continue the method of separation of variables for the case of the Robin condition. The Robin condition means that we are solving $-X''=\lambda x$ with the boundary conditions

$$X'-a_0X=0\quad at x=0---(1)$$
$$X'+a_1X=0\quad at x=l---(2)$$

The two constants $a_0\ and\ a_1$ should be considered as given.

Notes

The physical reason they are written with opposite signs is that they correspond to radiation of energy if a_0 *and* a_1 are positive, absorption of energy if a_0 *and* a_1 are negative, and insulation if $a_0 = a_1 = 0$. This is the interpretation for a heat problem: See the discussion section 1.4 or Exercise 2.3.8. For the case of the vibrating string, the interpretation is that the string shares its energy with the endpoints if a_0 *anda*$_1$ are positive, whereas the string gains some energy from the endpoints if a_0 *anda*$_1$ are negative:

The mathematical reason for writing the constants in this way is that the unit outward normal n for the interval $0 \le x \le l$ points to the left at $x = 0(n = -1)$ and to the right at $x = l(n = +1)$. Therefore, we expect that the nature of the eigenfunctions might depend on the signs of the two constants in opposite ways.

Check your progress

1. Explain the method of separation of variables for the case of the Robin condition.

5.5 POSITIVE EIGEN VALUES

Our task now is to solve the ODE $-X'' = \lambda X$ with the boundary conditions (1), (2). First let's look for the positive eigenvalues

$$\lambda = \beta^2 > 0.$$

As usual, the solution of the ODE is

$$X(x) = C\cos\beta x + D\sin\beta x \text{------(3)}$$

So that

$$X'(x) \pm aX(x) = (\beta D \pm aC)\cos\beta x + (-\beta C \pm aD)\sin\beta x.$$

At the left end $x = 0$ we require that

$$0 = X'(0) - a_0 X(0) = \beta D - a_0 C. \text{------}(4)$$

So we can solve for D in terms of C. At the right end $x = 1$ we require that

$$0 = (BD + a_l C)\cos\beta l + (-\beta C + a_l D)\sin\beta l. \text{-----}(5)$$

Messy as they may look, equations (4) and (5) are easily solved since thry are equivalent to the matrix equation

$$\begin{pmatrix} -a_0 & \beta \\ a_l \cos\beta l - \beta \sin\beta l & \beta \cos\beta_l + \mathrm{a}_l \sin\beta l \end{pmatrix} \begin{pmatrix} \mathrm{C} \\ \mathrm{D} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{------}(6)$$

Therefore, substituting for D, we have

$$0 = (a_0 C + a_l c)\cos\beta l + \left(-\beta C + \frac{a_l a_0}{\beta}\right)\sin\beta l. \text{-------}(7)$$

We don't want the trivial solution $C = 0$. We divide by $C\cos\beta l$ and multiply by β to get

$$(\beta^2 - a_0 a_1)\tan\beta l = (a_0 + a_1)\beta. \text{--------}(8)$$

Any root $\beta > 0$ of this "algebraic" equation would give us an eigenvalue $\lambda = \beta^2$.

What would be the corresponding eigenfunction? It would be the above $X(x)$ with the required relation between $C \text{ and } D$, namely,

$$X(x) = C\left(\cos\beta x + \frac{a_0}{\beta}\sin\beta x\right) \text{-------}(9)$$

For any $C \neq 0$. By the way, because we divided by $\cos\beta l$, there is the exceptional case when $\cos\beta l = 0$; it would mean by (7) that $\beta = \sqrt{a_0 a_l}.$

Our next task is to solve (8) for β. This is not so easy, as there is no simple formula. One way is to calculate the roots numerically, say by Newton's method.

Another way is by graphical analysis, which, instead of precise numerical values, will provide a lot of qualitative information.

This is what we'll do. It's here where the nature of $a_0 \, and \, a_l$ come into play. Let us rewrite the eigenvalue equation (8) as

$$\tan \beta l = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l}. \text{------}(10)$$

Our method is to sketch the graphs of the tangent function $\mathrm{y} = \tan \beta l$ and the rational function $y = {(a_0 + a_l)\beta}\big/{(\beta^2 - a_0 a_l)}$ as function of $\beta > 0$ and to find their points of intersection. What the rational function looks like depends on the constants $a_0 \, and \, a_l$.

Case 1 In Figure 1 is pictured the case of radiation at both ends : $a_0 > 0$ and $a_l > 0$.

Each of the points of intersection $(for \beta > 0)$ provides an eigenvalue $\lambda_n = \beta_n^2$. The results depend very much on the $a_0 \, and \, a_l$.

The exceptional situation mentioned above, when $\cos \beta l = 0 \, and \, \beta \sqrt{a_0 a_l},$ will occur when the graphs of the tangent function and the rational function "intersect at infinity."

No matter what they are, as long as they are both positive, the graph clearly shows that

$$n^2 \frac{\pi^2}{l^2} < \lambda_n < (n+1)^2 \frac{\pi^2}{l^2} \quad (n = 0, 1, 2, 3, \ldots). \text{-----}(11)$$

Furthermore,

$$\lim_{n \to \infty} \beta_n - n\frac{\pi}{l} = 0, \text{--------}(12)$$

Which means that the larger eigenvalues get relatively closer to ${n^2 \pi^2}\big/{l^2}$ (see Exercise 19). You may compare this to the case $a_0 = a_l = 0,$ the Neumann problem, where they are all exactly equal to ${n^2 \pi^2}\big/{l^2}$.

Case 2 the case of absorption at $x = 0$ and radiation at $x = l$, but more radiation than absorption, is given by the conditions

$$a_0 < 0, a_l > 0, a_0 + a_l > 0. \text{-----}(13)$$

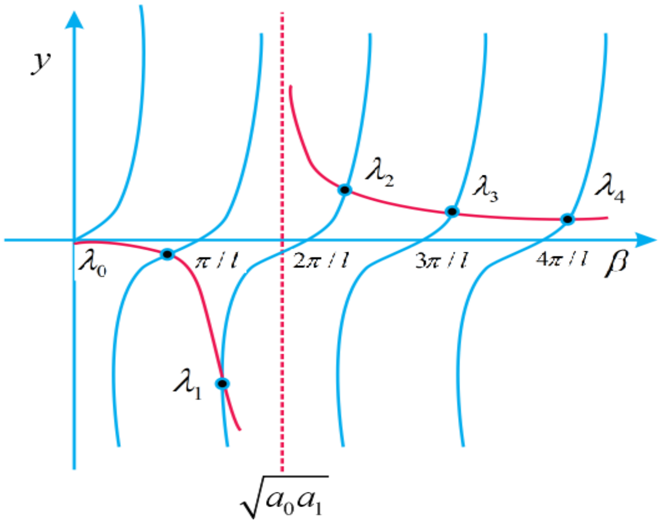


Figure 1

Then the graph looks like Figure 2 or 3, depending on the relative sizes of a_0 *and* a_l. Once again we see that (11) and (12) hold, except that in Figure 2 there is no eigenvalue λ_0 in the interval $\left(0, \pi^2/l^2\right)$.

There is an eigenvalue in the interval $\left(0, \pi^2/l^2\right)$. only if the rational curve crosses the first branch of the tangent curve. Since the rational curve has only a single maximum, this crossing can happen only if the slope of the rational curve is greater than the slope of the tangent curve at the origin. Let's

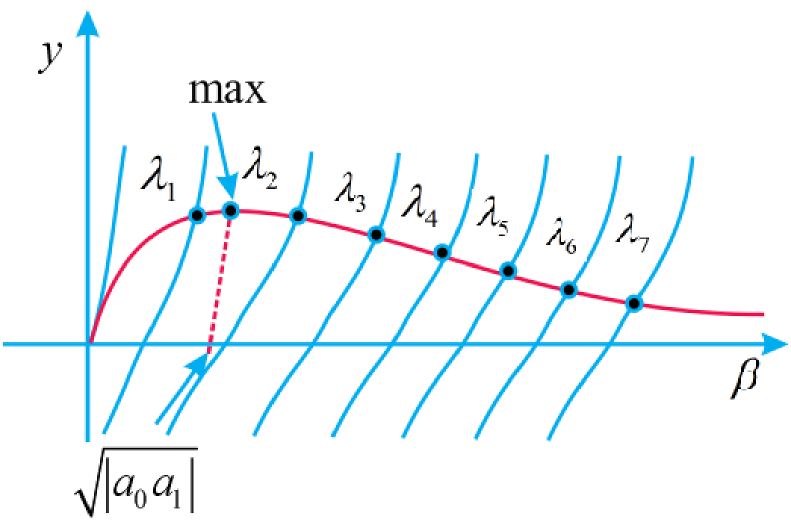


Figure 2

Notes

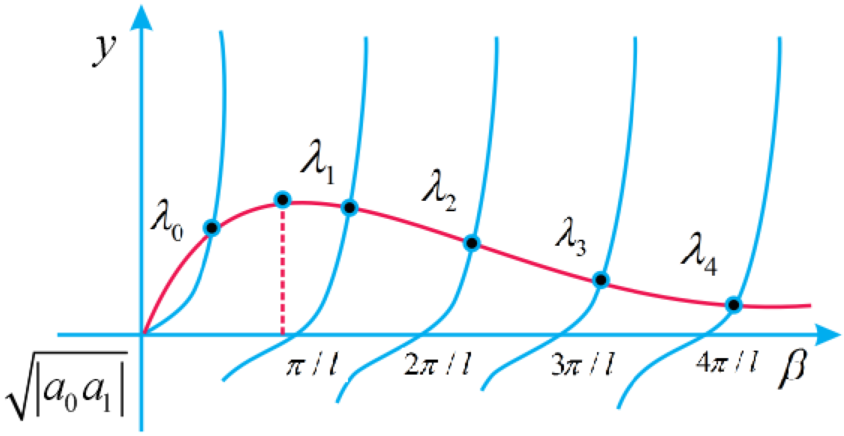


Figure-3

Calculate these two slopes. A direct calculation shows that the slope $dy/d\beta$ of the rational curve at the origin is

$$\frac{a_0 + a_l}{-a_0 a_l} = \frac{a_l - |a_0|}{a_l |a_0|} > 0$$

Because of (13). On the other hand, the slope of the tangent curve $y = \tan l\beta$ at the origin is $l \sec^2(l0) = l.$ thus we reach the following conclusion. In case

$a_0 + a_l > -a_0 a_l l$ ------(14)

(Which means "much more radiation than absorption"), the rational curve will start out at the origin with a greater slope than the tangent curve and the two graphs must intersect at a point in the interval $\left(0, \pi/2l\right)$. therefore, we conclude that in Case 2 there is an eigenvalue $0 < \lambda_0 < \left(\pi/2l\right)^2$ if and only if (14) holds.

5.6 ZERO EIGEN VALUE

In Exercise 2 it shown that there is a zero eigenvalue if and only if

$$a_0 + a_l = -a_0 a_l l. \text{------------}(15)$$

Notice that (15) can happen only if $a_0 \, or \, a_l$ is negative and the interval has exactly a certain length or else $a_0 = a_l = 0$.

NEGATIVE EIGENVALUE

Now let's investigate the possibility of a negative eigenvalue. This is a very important question; see the discussion at the end of this section.

To avoid dealing with imaginary numbers, we set

$$\lambda = -\gamma^2 < 0$$

And write the solution of the differential equation as

$$X(x) = C\cosh\gamma x + D\sinh\gamma x.$$

(As alternative form, which we used at the end of section 5.1, is $Ae^{\gamma x} + Be^{-\gamma x}$.) The boundary conditions, much as before , lead to the eigenvalue equation

$$\tanh\gamma l = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}.\text{--------(16)}$$

(Verify it!) So we look for intersections of these two graphs [on the two sides of (16)] for $\gamma > 0.$ Any such point of intersection would provide a negative eigenvalue $\lambda = -\gamma^2$ and a corresponding eigenfunction

$$X(x) = \cosh\gamma x + \frac{a_0}{\gamma}\sinh\gamma x.\text{------(17)}$$

Several different cases are illustrated in Figure 4. Thus in Case 1, of radiation at both ends, when $a_0\, and\, a_l$ are both positive, there is no intersection and so no negative eigenvalue.

Case 2, the situation with more radiation than absorption $(a_0 < 0, a_l > 0, a_0 + a_l > 0)$, is illustrated by the two solid (14) and dashed (18) curves. There is wither one intersection or none, depending on the slopes at the origin. The slope of the tan h curve is l , while the slope of the rational curve is

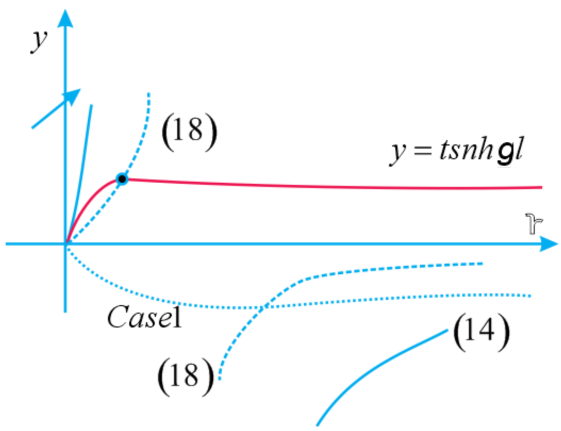


Figure 4

$-\left.(a_0 + a_l)\middle/(a_0 a_l)\right. > 0.$ If the last expression is smaller than l, there is

Notes

an intersection; otherwise, there isn't. So out conclusion in Case 2 is as follows.

Let $a_0 < 0 \, and \, a_l > -a_0$. If

$$a_0 + a_l < -a_0 \, \mathrm{a}_l \mathrm{l}.\text{-------}(18)$$

Then there exists exactly one negative eigenvalue, which we'll call $\lambda_0 < 0$. If (140 holds, then there is no negative eigenvalue. Notice how the "missing" positive eigenvalue λ_0 in case 918) now makes its appearance as a negative eigenvalue! Furthermore, the zero eigenvalue is the borderline case (15); therefore, we use the notation $\lambda_0 = 0.$ in the case of (15).

SUMMARY:

We summarize the various cases as follows:

Case 1: Only positive eigenvalues.

Case 2 With (14): Only positive eigenvalues.

Case 2 With (15): Zero is an eigenvalue, all the rest are positive.

Case 2 With (18): one negative eigenvalue, all the rest are positive.

In any case, that is, for any values for $a_0 \, and \, a_l$, there are no complex, nonreal, eigenvalues. This fact can be shown directly as before but will also be shown by a general, more satisfying, argument in Section 5.3. Furthermore, there are always an infinite number of possible eigenvalues, as is clear from (10). In fact, the tangent function has an infinite number of branches. The rational function on the right side of (100 always goes from the origin to the β axis as $\beta \to \infty$ and so must cross each branch of the tangent except possibly the first one.

For all these problems it is critically important to find all the eigenvalues. If even one of them were missing, there would be initial data for which we could not solve the diffusion or wave equations. This will become clearer in Chapter 5. Exactly how we enumerate the eigenvalues, that is, whether we call the first one $\lambda_0 \, or \, \lambda_l \, or \, \lambda_5 \, or \, \lambda_{-2}$, is not important. It is convenient, however, to number them in a consistent way. In the examples presented above we have numbered them in a way that neatly exhibits their dependence on $a_0 \, and \, a_l$.

What Is the Grand Conclusion for the Robin BCs? As before, we have an expansion

$$u(x,t) = \sum_n T_n(t) X_n(x), \text{ ------(19)}$$

Where $X_n(x)$ are the eigen functions and where

$$T_n(t) = \begin{cases} A_n e^{-\lambda_n kt} & \text{for diffusion} \\ A_n \cos\left(\sqrt{\lambda_n ct}\right) + B_n \sin\left(\sqrt{\lambda_n ct}\right) & \text{for waves.} \end{cases} \text{ --------}$$

(20)

Example 1.

Let $a_0 < 0 < a_0 + a_l < -a_0 a_l l$, which is Case 2 with (18). Then the grand conclusion takes the following explicit form. As we showed above, in this case there is exactly one negative eigenvalue $\lambda_0 = -\gamma_0^2 < 0$ as well as a sequence of positive ones $\lambda_n = +\beta_n^2 > 0 \; for \, n = 0,1,2,3,....$ The complete solution of the diffusion problem

$$\begin{aligned} &u_t = ku_{xx} && for\, 0 < x < l,\ 0 < t < \infty \\ &u_x - a_0 u = 0 \;\; forx = 0,\ u_x + a_l u = 0 && forx = l \\ &u \neq 0 && fort = 0 \end{aligned}$$

Therefore is

$$\begin{aligned} u(x,t) = A_0 e^{+\gamma_0^2}\left(\cosh \gamma 0x + \frac{a_0}{\gamma_0}\sinh \gamma_0 x\right) \\ + \sum_{n=1}^{\infty} A_n e^{-\beta_n^2 kt}\left(\cos \beta_n x + \frac{a_0}{\beta_n}\sin \beta_n x\right). \end{aligned} \text{ ---------(21)}$$

The conclusion (21) has the following physical interpretation if, say, $u(x,t)$ is the temperature in a rod of length l. We have taken taken the case when energy is supplies at $x = 0$ (absorption of energy by the rod, heat flux goes into the rod at its left end) and when energy is radiated from the right end (the heat flux goes out.) For a given length l and a given radiation $a_l > 0$, there is a negative eigenvalue $\left(\lambda_0 = -\gamma_0^2\right)$ if and only if the absorption is great enough $\left[|a_0| > a_l \big/ (1 + a_l l)\right]$. Such a large absorption coefficient allows the temperature to build up to large up to large values, as we see from the

Notes

expansion (21). In fact, all the terms get smaller as time goes on, except the first one, which grows exponentially due to the factor $e^{+\gamma_0^2 kt}$. So the rod gets hotter and hotter (unless $A_0 = 0$, which could only happen for very special initial data.)

If, on the other hand, the absorption is relatively small [That is, $\left[|a_0| > {a_l}/{(1+a_l l)}\right]$.then all the eigenvalues are positive and the temperature will remain bounded and will eventually decay to zero. Other interpretations of this sort are left for the exercises.

For the wave equation, a negative eigenvalue $\lambda_0 = -\gamma_0^2$ would also lead to exponential growth because the expansion for $u(x,t)$ would contain the term

$$\left(A_0 e^{\gamma_0 ct} + B_0 e^{-\gamma_0 ct}\right) X_0(x).$$

This term comes from the usual equation $-T'' = \lambda c^2 T = -(\gamma_0 c)^2 T$ for the temporal part of a separated solution.

EXERCISE

1. Find the eigenvalues graphically for the boundary conditions

$$X(0) = 0, \qquad X(l) + aX(l) = 0.$$

Assume that $a \neq 0$.

2. Consider the eigenvalue problem with Robin BCs at both ends:

$$-X'' = \lambda X$$
$$X'(0) - a_0 X(0) = 0, \qquad X'(l) + a_l X(l) = 0.$$

(a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$.

(b) Find the eigenfunctions corresponding to the zero eigenvalue. (Hint: First solve the ODE for $X(x)$.The solutions are not sine's or cosines.)

3. Derive the eigenvalue equation (16) for the negative eigenvalues $\lambda = -\gamma^2$ and the formula (17) for the eigenfunctions.

4. Consider the Robin eigenvalue problem. If

$$a_0 < 0, \; a_l < 0 \quad and \quad -a_0 - a_l < a_0 a_l l,$$

Show that there are two negative eigenvalues. This case may be called "substantial absorption at both ends." (Hint: Show that the rational curve $y = -\left(a_0 + a_l\right)\gamma \big/ (\gamma^2 + a_0 a_l)$ has a single maximum and crosses the line $y = 1$ in two places. Deduce that it crosses the tanh curve in two places.)

5. If $a_0 = a_l = a$ in the Robin problem, show that:

(a) There are no negative eigenvalues if $a \geq 0,$ there is one if $-2/l < a < 0,$ and there are two if $a < -2/l$.

(b) Zero is an eigenvalue if and only if $a = 0\, or\, a = -2/l$.

6. If $a_0 = a_l = a$, show that as $a \to +\infty,$ the eigenvalues tend to the eigenvalues of the Dirichlet problem. That is,

$$\lim_{a\to\infty}\left\{\beta_n\left(a\right) - \frac{\left(n+1\right)\pi}{l}\right\} = 0,$$

Where $\lambda_n\left(a\right) = \left[\beta_n\left(a\right)\right]^2$ is the $\left(n+1\right)st$ eigenvalue.

7. Consider again Robin BCs at both ends for arbitrary $a_0\, and\, a_l$.

(a) In $a_0\, a_l$ plane sketch the hyperbola $a_0 + a_l = -a_0 a_l l.$ Indicate the asymptotes. For $(a_0\, a_l)$ on this hyperbola, zero is an eigenvalue, according to Exercise 2(a).

(b) Show that the hyperbola separates the whole plane into three regions, depending on whether there are two, one, or no negative eigenvalues.

(c) Label the directions of increasing absorption and radiation on each axis. Label the point corresponding to Neumann BCs.

(d) Where in the plane do the Dirichlet BCs belong?

8. On the interval $0 \leq x \leq 1$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X$$
$$X'\left(0\right) + X\left(0\right) = 0 \quad and \quad X\left(1\right) = 0$$

(absorption at one end and zero at the other).

(a) Find an eigenfunction with eigenvalue zero. Call it $X_0\left(x\right)$.

(b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

Notes

(c) Show graphically from part (b) that there are an infinite number of positive eigenvalues.

(d) Is there a negative eigenvalue?

9. Consider the unusual eigenvalue problem

$$-\upsilon_{xx} = \lambda \upsilon \qquad for\, 0 < x < l$$
$$\upsilon_x(0) = \upsilon_x(l) = \frac{\upsilon(l) - \upsilon(0)}{l}.$$

(a) Show that $\lambda = 0$ is a double eigenvalue.

(b) Get an equation for the positive eigenvalues $\lambda > 0$.

(c) Letting $\gamma = \frac{1}{2} l \sqrt{\lambda},$ reduce the equation in part (b) to the equation

$\gamma \sin \gamma \cos \gamma = \sin^2 \gamma.$

(d) Use part (c) to find half of the eigenvalues explicitly and half of them graphically.

(e) Assuming that all the eigenvalues are nonnegative, make list of all the eigenfunctions.

(f) Solve the problem $u_t k u_{xx}$ $for\, 0 < x < l,$ with the BCs given above, and with $u(x,0) = \phi(x).$

(g) Show that, as $t \to \infty, \lim(ux,t) = A + Bx$ for some constants A,B, assuming that you can take limits term by term.

10. Consider a string that is fixed at the end $x = 0$ and is free at the end $x = l$ except that a load (weight) of given mass is attached to the right end.

(a) Show that it satisfies the problem

$$u_{tt} = c^2 u_{xx} \qquad for\, 0 < x < l$$
$$u(0,t) = 0 \quad u_{tt}(l,t) = -k u_x(l,t)$$

For some constant k .

(b) What is the eigenvalue problem in this case?

(c) Find the equation for the positive eigenvalues and find the eigenfunctions.

12. Find the positive eigenvalues and the corresponding eighen functions of the fourth-order operator $+d^4 / dx^4$ with the four boundary conditions

$$X(0) = X(l) = X''(l) = 0$$

13. Solve the fourth-order eighen value problem $X'''' = \lambda X \ in \ 0 < x < l,$ with the four boundary conditions

$$X(0) = X'(0) = X(l) = X'(l) = 0,$$

Where $\lambda > 0$. (Hint : First solve the fourth-order ODE.)

14. A tuning fork may be regarded as a pair of vibrating flexible bars with a certain degree of stiffness. Each such bar is calmped at one end and is approximately modeled by the fourth-order PDE $u_u + c^2 u_{xxxx} = 0.$ It has initial conditions as for the wave equation. Let's say that on the end x=0 it is clamped (fixed), meaning that it satisfies.

$u(0,t) = u_x(0,t) = 0.$ On the other end $x = l$ it is free, meaning that it satisfies $u_{xx}(l,t) = u_{xxx}(l,t) = 0.$ Thus there are a total of four boundary conditions, two at each end.

a) Separate the time and space variables to get the eigenvalue problem $X^{nn} = \lambda X$.

b) Show that zero is not an eigenvalue.

c) Assuming that all the eighenvalue are positive, write them as $\lambda = \beta^4$ and find the equation for β.

d) Find the frequencies of vibration.

e) Compare your answer in part (d) with the overtones of the vibrating sting by looking at the ratio β_2^2 / β_1^2 .Explain why you hear an almost pure tone when you listen to a tuning fork.

Check your progress

2. Explain about positive Eigen values.

5.7 LET US SUM UP

In this we have discussed about Separation of variables, the Dirichlet condition, The Neumann condition, The Robin condition, Positive eigen values, Negative eigenvalues.Zero eigen values. Eigenvalues and the functions. The Neumann boundary conditions.

A “mixed: boundary condition would be Dirichlet at one end and Neumann at the other. In any case, that is, for any values for a_0 *and* a_l, there are no complex, nonreal, eigenvalues.

5.8 KEY WORDS

1. Homogeneous Dirichlet conditions for the wave equation
2. Schtrodinger’s equation
3. Neumann and Robin boundary conditions
4. Method of separation of variables for the case of the Robin condition.
5. Zero is an eigenvalue, all the rest are positive.
6. One negative eigenvalue, all the rest are positive.

5.9 QUESTIONS FOR REVIEW

1. Discuss about separation of variables, the Dirichlet Condition
2. Discuss about the Neumann condition
3. Discuss about the Robin condition
4. Discuss about Positive eigen values

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5.11 ANSWERS TO CHECK FOR YOUR PROGRESS

1. See section 5.3
2. See section 5.5
3. See section 5.6

UNIT-6 HARMONIC FUNCTIONS

STRUTURE

6.0 OBJECTIVE

In this unit we will learn and understand about Laplace equation, Maximum principle,

Rectangles and cubes, Poissons's formula, circles, wedges and annuli.

6.1 INTRODUCTION

This chapter is devoted to the Laplace equation. We introduce two of its important properties, the maximum principle and the rotational invariance. Then we solve equation in series form in rectangles, circles, and related shapes. The case of a circle leads to the beautiful Poisson formula.

6.2 LAPLACE'S EQUATION

If a diffusion or wave process is stationary (independent of time), then $u_t \equiv 0$ and $u_t \equiv 0$. Therefore, both the diffusion and the wave equations reduce to the Laplace equation:

$$\begin{aligned}
&u_{xx} = 0 && in\,one\,\dim ension\\
&\nabla.\nabla_u = \nabla u = u_{xx} + u_{yy} = 0 && in\,two\,\dim ensions\\
&\nabla.\nabla_u = \Delta u = u_{xx} + u_{yy} + u_{zz} = 0 && in\,three\,\dim ensions
\end{aligned}$$

A solution of the Laplace equation is called a harmonic function.

In one dimension, we have simply $u_{xx} = 0,$ so the only harmonic functions in one dimension are $u(x) = A + Bx.$ But this is so simple that it hardly gives us a clue to what happens in higher dimensions.

The inhomogeneous version of Laplace's equation

$$\Delta u = f$$

With f a given function, is called Poisson's equation.

Besides stationary diffusions and waves, some other instances of Laplace's and Poisson's equations include the following.

1. Electrostatics: From Maxwell's equations, one has curl $E = 0$ and div $E = 4\pi p,$ where p is the charge density. The first equation implies $E = -grad\,\phi$ for a scalar function ϕ (called the electric potential). Therefore,

$$\Delta\phi = div\left(grad\phi\right) = -div\,E = -4\pi p,$$

Which is Poisson's equation $\left(with\ f = -4\pi p\right).$

2. Steady fluid flow. Assume that the flow is irrational (no eddies) so that $curl\,v = 0, where\,v = v\left(x, y, z\right)$ is the velocity at the position $\left(x, y, z\right)$, assumed independent of time. Assume that the fluid is incompressible (e.g., water) and that there are no sources or sinks. Then $div\,v = 0.$ Hence $v = -grad\phi\ for\ some\,\phi$ (called the velocity potential) and $\Delta\phi = -div\,v = 0,$ which is Laplace's equation.

3. Analytic functions of a complex variable. Write $z = x + iy$ and

$$f(z) = u(z) + i\upsilon(z) = u(x+iy) + i\upsilon(x+iy),$$

Where $u \, and \, \upsilon$ are real- valued functions. An analytic function is one that is expressible

as a power series in z. This means that the powers are not $x^m y^n \, but \, z^n = (x+iy)^n$. Thus

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$(a_n \, complex \, constants)$. That is,

$$u(x+iy) + i\upsilon(x+iy) = \sum_{n=0}^{\infty} a_n (x+iy)^n.$$

Formal differentiation of this series show that

$$\frac{\partial u}{\partial x} = \frac{\partial \upsilon}{\partial y} \, and \, \frac{\partial u}{\partial y} = -\frac{\partial \upsilon}{\partial x}$$

(see Exercise 1). These are the Cauchy-Riemann equations. If we differentiate them, we

find that

$$u_{xx} = \upsilon_{yx} = \upsilon_{xy} = -u_{yy},$$

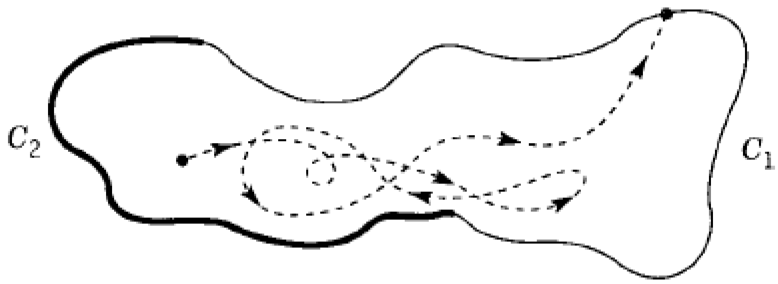
So that $\Delta u = 0.$ Similarly $\Delta \upsilon = 0,$ where Δ is the two-dimensional laplacian. Thus the

real and imaginary parts of an analytic function are harmonic.

4. Brownian motion. Imagine Brownian motion in a container D. This means that particles inside D move completely randomly until they hit the boundary, when they stop. Divide the boundary arbitrarily into two pieces, $C_1 \, and \, C_2$ (see Figure 1). Let $u(x,y,z)$ be the probability that a particle that begins at the point (x,y,z) stops at some point of C_1. Then it can be deduced that

$$\Delta u = 0 \, in \, D$$
$$u = 1 \, on \, C_1 \qquad u = 0 \, on \, C_2.$$

Thus u is the solution of a Dirichlet problem.



As we discussed in previous the basic mathematical problem is to solve Laplace's or Poisson's equation in a given domain D with a condition on $bdy\, D$.

$$\Delta u = f \; in\, D$$

$$u = h \quad or \quad \frac{\partial u}{\partial n} = h \;\; or \;\; \frac{\partial u}{\partial n} + au = h \;\; on\, bdy\, D.$$

In one dimension the only connected domain is an interval $\{a \le x \le b\}$. We will see that what is intersecting about the two- and three-dimensional cases is the geometry.

Check your prorgress

1. Explain about Laplace equation.

6.3 MAXIMUM PRINCIPLE

We begin our analysis with the maximum principle, which is easier for Laplace's equation than for the diffusion equation. By an open set we mean a set that includes none of its boundary points.

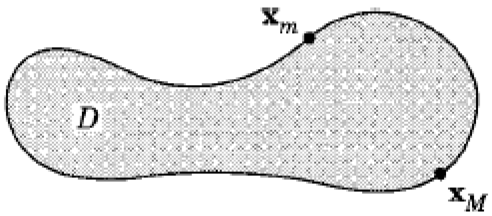
Maximum Principle. Let D be a connected bounded open set (in either two- or three-dimensional space). Let either $u(x,y) or\, u(x,y,z)$ be a harmonic function in D that is continuous on $\bar{D} = D \cup (bdy\, D)$. Then the maximum and minimum values of u are attained on $bdy\, D$ and nowhere inside $(unless\, u \equiv cons\tan t)$.

In other words, a harmonic function is its biggest somewhere on the boundary and its smallest somewhere else on the boundary.

Notes

To understand the maximum principle, let us use the vector shorthand $x = (x, y)$ in two dimensions or $x = (x, y, z)$ in three dimensions. Also, the radial coordinate is written as $|x| = (x^2 + y^2)^{1/2}$ or $|x| = (x^2 + y^2 + z^2)^{1/2}$. The maximum principle asserts that there exist points X_M and X_M $on\, bdy\, D$ such that

$$u(X_m) \le u(X) \le u(X_M)$$



for all $X \in D$ (see Figure 2). Also, there no points inside D with this property $(unless\, u \equiv cons\tan t)$. There could be several such points on the boundary.

The idea of the maximum principle is as follows, in two dimensions, say. At a maximum point inside D, if there were one, we'd have $u_{xx} \le 0\, and\, u_{yy} \le 0.$ At most maximum points, $u_{xx} < 0\, and\, u_{yy} < 0.$ So we'd get a contradiction to Laplace's equation. However, since it is possible that $u_{xx} = 0 = u_{yy}$ at a maximum point, we have to work a little harder to get a proof.

Here we go. Let $\in > 0.$ Let $\upsilon(X) = u(X) + \in |X|^2$. Then, still in two dimensions, say,

$$\Delta\upsilon = \Delta u + \in \Delta(x^2 + y^2) = 0 + 4 \in > 0\ in\, D.$$

But $\Delta\upsilon = \upsilon_{xx} + \upsilon_{yy} \le 0$ at an interior maximum point, by the second derivative test in calculus! Therefore, $\upsilon(X)$ has no interior maximum in D.

Now $\upsilon(X)$, being a continuous function, has to have a maximum somewhere in the closure $\bar{D} = D \cup bdy\, D.$ Say that the maximum of $\upsilon(X)$ is attained at $X_0 \in bdy\, D.$ Then, for all $X \in D.$

$$u(X) \le \upsilon(X) \le \upsilon(X_0) = u(X_0) + \in |X_0|^2 \le \max_{bdy\, D} u + \in l^2,$$

Where l is the greatest distance from $bdy\, D$ to the origin. Since this is true for any $\in > 0,$ we have

$$u(X) \le \max_{bdy\, D} u \qquad for\, all\, X \in D.$$

Now this maximum is attained at some point

$X_M \in bdy\, D. So\, u(X) \le u(X_M)\ for\, all\ X \in \bar{D},$ which is the desired conclusion. The existence of a minimum point x_m is similarly demonstrated. (The absence of such points inside D will be proved by a different method in Section 6.3.)

UNIQUENESS OF THE DIRICHLET PROBLEM

To prove the uniqueness, suppose that

$$\begin{aligned} &\Delta u = f \quad inD \quad \Delta \upsilon = f \quad in\, D \\ &u = h \quad onbdy\, D \quad \upsilon = h \quad onbdyD. \end{aligned}$$

We want to show that $u \equiv \upsilon\, in\, D.$ So we simply subtract equations and let $w = u - \upsilon.$ Then $\Delta w = 0\, in\, D$ on $bdy\, D.$ By the maximum principle

$$0 = w(X_m) \le w(X) \le w(X_M) = 0 \qquad for\, all\ X \in D.$$

Therefore, both the maximum and minimum of $w(X)$ are zero. This means that $w \equiv 0\, and\, u \equiv \upsilon.$

INVARIANCE IN TWO DIMENSIONS

The Laplace equation is invariant under all rigid motions. A rigid motion in the plane consists of translations and rotations. A translation in the plane is a transformation

$$x' = x + a \qquad y' = \mathrm{y} + \mathrm{b}.$$

Notes

Invariance under translations means simply that $u_{xx}+u_{yy}=u_{x'x'}+u_{y'y'}$.

A rotation in the plane through the angle α is given by

$$\begin{aligned} x' &= x\cos\alpha + y\sin\alpha \\ y' &= -x\sin\alpha + \cos\alpha. \end{aligned}$$

By the chain rule we calculate

$$\begin{aligned} u_x &= u_{x'}\cos\alpha - u_{y'}\sin\alpha \\ u_y &= u_{x'}\sin\alpha + u_{y'}\cos\alpha \\ u_{xx} &= \left(u_{x'}\cos\alpha - u_{y'}\sin\alpha\right)_{x'}\cos\alpha - \left(u_{x'}\cos\alpha - u_{y'}\sin\alpha\right)_{y'}\sin\alpha \\ u_{yy} &= \left(u_{x'}\sin\alpha + u_{y'}\cos\alpha\right)_{x'}\sin\alpha + \left(u_{x'}\sin\alpha + u_{y'}\cos\alpha\right)_{y'}\cos\alpha. \end{aligned}$$

Adding, we have

$$\begin{aligned} u_{xx}+u_{yy} &= \left(u_{x'x'}+u_{y'y'}\right)\left(\cos^2\alpha+\sin^2\alpha\right)+u_{x'y'}.(0) \\ &= u_{x'x'}+u_{y'y'}. \end{aligned}$$

This proves the invariance of the Laplace operator. In engineering the laplacian Δ is a model for isotropic physical situations, in which there is no preferred direction.

The rotational invariance suggests that the two-dimensional laplacian

$$\Delta_2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

Should take a particularly simple form in polar coordinates. The transformation

$$x = r\cos\theta \qquad y = r\sin\theta$$

Has the jacobian matrix

$$\wp = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

With the inverse matrix

$$\wp^{-1} = \begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial \theta}{\partial x} \\ \dfrac{\partial r}{\partial y} & \dfrac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \dfrac{-\sin\theta}{r} \\ \sin\theta & \dfrac{\cos\theta}{r} \end{pmatrix}$$

(Beware, however, that $\partial r/\partial x \neq \left(\partial x/\partial r\right)^{-1}$. So by the chain rule we have

$$\frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta},$$
$$\frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

These operators are squared to give

$$\frac{\partial^2}{\partial x^2}=\left[\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]^2$$
$$=\cos^2\theta\frac{\partial^2}{\partial r^2}-2\left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial^2}{\partial r\partial\theta}$$
$$+\frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial}{\partial r}$$
$$\frac{\partial^2}{\partial y^2}=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)^2$$
$$=\sin^2\theta\frac{\partial^2}{\partial r^2}+2\left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial^2}{\partial r\partial\theta}$$
$$+\frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial}{\partial r}.$$

(The last two terms come from differentiation of the coefficients.) Adding these operators, we get (1o and behold!)

$$\Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

It is also natural to look for special harmonic functions that themselves are rotationally invariant. In two dimensions this means that we use polar coordinates (r,θ) and look for solutions depending only on r. Thus by (5)

$$0=u_{xx}+u_{yy}=u_{rr}+\frac{1}{r}u_r$$

If u does not depend on θ. This ordinary differential equation is easy to solve:

$$\left(ru_r\right)_r 0,\quad rurc_1,\quad u=c_1\log\mathrm{r}+c_2.$$

The function $\log\mathrm{r}$ will play a central role later.

INVARIANCE IN THREE DIMENSIONS

Notes

The three- dimensional laplacian is invariant under all rigid motions in space. To demonstrate its rotational invariance we repeat the preceding proof using vector-matrix notation. Any rotation in three dimensions is given by

$$X' = BX,$$

Where B is an orthogonal matrix $\left({}^{t}BB = B^{t}B = 1 \right)$. The laplacian is $\Delta u = \sum_{i=1}^{3} u_{ii} = \sum_{i,j=1}^{3} \delta_{ij} u_{ij}$ where the subscripts of u denote partial derivatives. Therefore,

$$\begin{aligned} \Delta u &= \sum_{k,l} \left(\sum_{i,j} b_{ki} \delta_{ij} b_{lj} \right) uk'l' = \sum_{k,l} \delta_{kl} u_{k'l'} \\ &= \sum_{k} u_{k'k'} \end{aligned}$$

Because the new coefficient matrix is

$$\sum_{i,j} b_{ki} \delta_{ij} b_{ij} = \sum_{i} bib_{li} = \left(B'B \right)_{kl} = \delta_{kl}.$$

So in the primed coordinates Δu takes the usual form

$$\Delta u = u_{x'x'} + u_{y'y'} + u_{z'z'}.$$

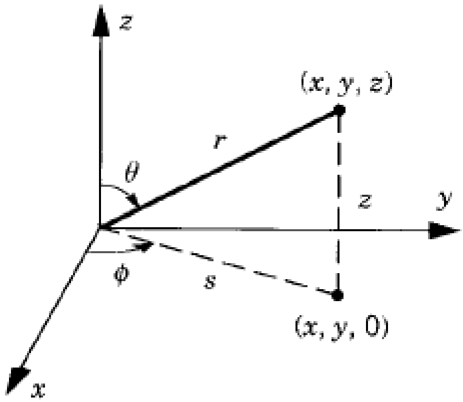
For the three-dimensional laplacian

$$\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y 2} + \frac{\partial^2}{\partial z^2}$$

it is natural to use spherical coordinates $(\mathrm{r}, \theta, \phi)$ (see Figure 3). We'll use the notation

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} = \sqrt{s^2 + z^2} \\ s &= \sqrt{x^2 + y^2} \\ x &= s\cos\phi \qquad z = r\cos\theta \\ y &= s\sin\phi \qquad s = r\sin\theta. \end{aligned}$$

(What out: In some calculus books the letters ϕ and θ are switched.) The calculation, which is a little tricky, is organized as follows. The chain of



Variables is $(x, yx, z) \to (s, \phi, z) \to (r, \theta, \phi)$. By the two-dimensional Laplace calculation, we have both

$$u_{zz} + u_{ss} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

and

$$u_{xx} + u_{yy} = u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.$$

We add these two equations, and cancel u_{ss}, to get

$$\begin{aligned}\Delta_3 &= u_{xx} + u_{yy} + u_{zz} \\ &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.\end{aligned}$$

In the last term we substitute $s^2 = r^2 \sin^2\theta$ and in the next-to-last term

$$\begin{aligned}u_s &= \frac{\partial u}{\partial s} = u_r\frac{\partial r}{\partial s} + u_0\frac{\partial \theta}{\partial s} + u_\phi\frac{\partial \phi}{\partial s} \\ &= u_r.\frac{s}{r} + u_\theta.\frac{\cos\theta}{r} + u_\phi.0.\end{aligned}$$

This leaves us with

$$\Delta_3 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right],$$

Which may also be written as?

$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

Notes

Finally, let's look for the special harmonic functions in three dimensions which don't change under rotations, that is, which depend only on r. By (7) they satisfy the ODE

$$0 = \Delta_3 u = u_{rr} + \frac{2}{r} u_r.$$

So $\left(r^2 u_r\right)_r = 0.$ It has the solutions $r^2 u_r = c_1.$ That is, $u = -c_1 r^{-1} + c_2.$ This important harmonic function

$$\frac{1}{r} = \left(x^2 + y^2 + z^2\right)^{-1/2}$$

Is the analog of the special two-dimensional function $\log = \left(x^2 + y^2\right)^{1/2}$ found before. Strictly speaking, neither function is finite at the origin. In electrostatics the function $u(X) = r^{-1}$ turns out to be the electrostatic potential when a unit charge is placed at the origin. For further discussion, see Section 12.2.

EXERCISE:

1. Show that a function which is a power series in the complex variable $x + iy$ must satisfy the Cauchy-Riemann equations and therefore Laplace's equation.

2. Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u,$ where k is a positive constant. $\left(H\,\text{int}: Substitute u = {}^{\upsilon}\!/_{r}.\right)$

3. Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} = k^2 u,$ where k is a positive constant. (Hint: Look up Bessel's differential equation in $[MF]$ or in section 10.5.)

4. Solve $u_{xx} + u_{yy} + u_{zz} = 0$ in the spherical shell $0 < a < r < b$ with the boundary conditions $u = A\, on\, r = a\, and\, u = B\, on\, r = b,$ where A and B are constants. (Hint: Look for a solution depending only on r.)

5. Solve $u_{xx} + u_{yy} = 1 in\, r < a$ with $u(x, y)$ vanishing on $r = a.$

6. Solve $u_{xx} + u_{yy} = 1$ in the annulus $a < r < b.$ with $u(x, y)$ vanishing on both parts of the boundary $r = a\, r = b.$

7. Solve $u_{xx} + u_{yy} = 1$ in the spherical shell $a < r < b$ with $u = (x, y, z)$ vanishing on both the inner and outer boundaries.

8. Solve $u_{xx} + u_{yy} = 1$ in the spherical shell $a < r < b$ with $u = 0$ on $r = a\, and\, \partial u/\partial r = 0\, on\, r = b.$ Then let $a \to 0$ in your answer and interpret the result.

9. Prove the uniqueness of the Dirichlet problem $\Delta u = f\; in\, D, u = g$ on $bdy\, D$ by the energy method. That is after subtracting two solutions $w = u - \upsilon,$ multiply the Laplace equation for w by w itself and use the divergence theorem.

10. Show that there is no solution of

$$\Delta u = f\; in\, D, \frac{\partial u}{\partial n} = g\; on\, bdy\, D$$

in three dimensions, unless

$$\iiint_D f\, dx\, dy\, dz = \int \int_{bdy(D)} g\, dS.$$

(Hint: Integrate the equation.) Also show the analogue in one and two dimensions.

6.4 RECTANGLES AND CUBES

Special geometries can be solved by separating the variables.

(i)Look for separated solutions of the PDE.

(ii)Put in the homogeneous boundary conditions to get the eigenvalues. This is the step that requires the special geometry.

(iii)Sum the series.

(iv)Put in the inhomogeneous initial or boundary conditions.

Notes

It is important to do it in this order: homogeneous BC first, inhomogeneous BC last.

We begin with

$$\Delta_2 u = u_{xx} + u_{yy} = 0 \, in \, D,$$

Where D is the rectangle $\{0 < x < a, 0 < y < b\}$ on each of whose sides one of the standard boundary conditions is prescribed (inhomogeneous Dirichlet, Neumann, or Robin).

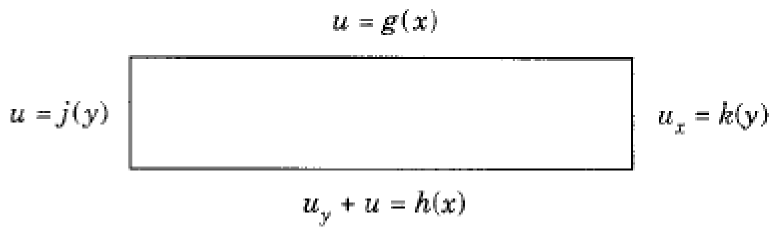


Figure 1

Example 1.

Solve (1) with the boundary conditions indicated in Figure 1. If we call the solution u with data $(g, h, j, k), then \, u = u_1 + u_2 + u_3 + u_4$ where u_1 has data $(g, 0, 0, 0), u_2 \, has \, data \, (0, h, 0, 0),$ and so on. For simplicity, let's assume that $h = 0,\ j = 0, and\ k = 0,$ so that we have Figure 2. Now we separate variable $u(x, y) = X(x).Y(y).$ We get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Hence there is a constant λ such that

$X'' + \lambda X = 0 \ for\ 0 \le x \le a \ and\ Y'' - \lambda Y = 0 \ for\ 0 \le x \le a \ and\ Y'' - \lambda Y = 0 \ and\ 0 \le y \le b.$

Thus $X(x)$ satisfies a homogeneous one-dimensional problem which we well know how to solve: $X(0) = X'(a) = 0.$ The solutions are

$$\beta_n^2 = \lambda_n = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{a^2} (n = 0, 1, 2, 3, \ldots)$$

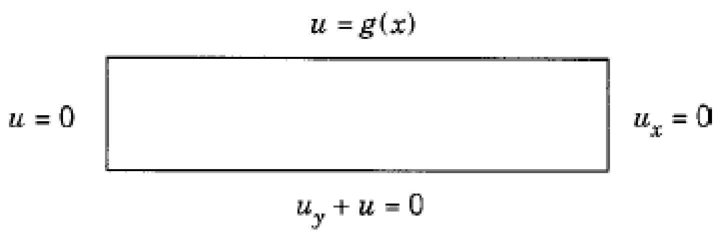
$$X_n(X) = \sin \frac{\left(n + \frac{1}{2}\right)\pi x}{a}.$$

Next we look at the y variable. We have

$$Y'' - \lambda Y = 0 \, with Y'(0) + Y(0) = 0.$$

(We shall save the inhomogeneous BCs for the last step.) From the previous part, we know that $\lambda = \lambda > 0$ for some $n.$ The Y equation has exponential solutions. As usual it is convention to write them as

$$Y(y) = A\cosh\beta_n y + B\sinh\beta_n y.$$



So $0 = Y'(0) + Y(0) = B\beta_n + A.$ Without losing any information we may pick $B = -1,$ so that $A = \beta_n.$ Then

$$Y(y) = \beta_n \cosh\beta_n y - sinh\beta_n y.$$

Because we're in the rectangle, this function is bounded. Therefore, the sum

$$u(x,y) = \sum_{n=0}^{\infty} A_n \sin\beta_n x\left(\beta_n \cosh\beta_n y - \sinh\beta_n y\right)$$

is a harmonic function in D that satisfies all three homogeneous BCs. The remaining BC is $u(x,b) = g(x).$ It requires that

$$g(x) = \sum_{n=0}^{\infty} A_n\left(\beta_n \cosh\beta_n b - \sinh\beta_n b\right).\sin\beta_n x$$

for $0 < x < a.$ This is simply a Fourier series in the eigenfunctions $\sin\beta_n X.$

By Chapter 5, the coefficients are given by the formula

$$A_n = \frac{2}{a}\left(\beta_n \cosh\beta_n b - \sinh\beta_n b\right)^{-1}\int_0^a g(x)\sin\beta_n x\,dx.$$

Notes

Example 2.

The same method works for a three-dimensional box $\{0 < x < a, 0 < y < b, 0 < z < c\}$ with boundary conditions on the six sides. Take Dirichlet conditions on a cube:

$$\begin{aligned}
&\Delta_3 u = u_{xx} + u_{yy} + u_{zz} = 0\, in\, D \\
&D = \{0 < x < \pi, 0 < y < \pi, 0 < z < \pi\} \\
&u(\pi, y, z) = g(y, z) \\
&u(0, y, z) = u(x, 0, z) = u(x, \pi, z) = u(x, y, 0) = u(x, y, \pi) = 0.
\end{aligned}$$

To solve this problem we separate variables and use the five homogeneous boundary conditions:

$$\begin{gathered}
u = X(x)Y(y)Z(z), \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0 \\
X(0) = Y(0) = Z(0) = Y(\pi) = Z(\pi) = 0.
\end{gathered}$$

Each quotient $X''/X, Y''/Y, and\, Z''/Z$ must be a constant. In the familiar way, we find

$$Y(y) = \sin my \qquad (m = 1, 2, \ldots)$$

and

$$Z(z) = \sin nz \qquad (n = 1, 2, \ldots),$$

$$so\, that \qquad X'' = (m^2 + n^2)X, \qquad X(0) = 0.$$

Therefore,

$$X(x) = A \sinh\left(\sqrt{m^2 + n^2}\, x\right).$$

Summing up, our complete solution is

$$u(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sinh\left(\sqrt{m^2 + n^2}\, x\right) \sin my \sin nz.$$

Finally, we plug in our inhomogeneous condition at $x = \pi$:

$$g(y, z) = \sum \sum A_{mn} \sinh\left(\sqrt{m^2 + n^2}\, \pi\right) \sin my \sin nz.$$

This is a double Fourier sine series in the variables $y\, and\, z$! Its theory is similar to that of the single series. In fact, the eigenfunctions

$\{\sin my.\sin nz\}$ are mutually orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$ (see Exercise 2). Their normalizing constants are

$$\int_0^{\pi}\int_0^{\pi}(\sin my \sin nz)^2\, dy\, dz = \frac{\pi^2}{4}.$$

Therefore,

$$A_{mn} = \frac{4}{\pi^2 \sinh\left(\sqrt{m^2+n^2}\,\pi\right)}\int_0^{\pi}\int_0^{\pi} g(y,z)\sin my \sin nz\, dy\, dz.$$

Hence the solutions can be expressed as the doubly infinite series (7) with the coefficients A_{mn}. The complete solution to Example 2 is (7) and (8). With such a series, as with a double integral, one has to be careful about the order of summation, although in most cases any order will give the correct answer.

EXERCISE:

1. Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < a, 0 < y < b$ with the following boundary conditions:

$$\begin{aligned} &u_x = -a \quad on\; x = 0 \qquad u_x = 0 \quad on\; x = a \\ &u_y = b \quad on\; y = 0 \qquad u_y = 0 \quad on\; y = b. \end{aligned}$$

(Hint: Note that the necessary condition of Exercise 6.1.11 is satisfied. A shortcut is to guess that the solution might be a quadratic polynomial in x and y.)

2. Prove that the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$.
3. Find the harmonic function $u(x,y)$ in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions:

$$u_y = 0 \quad for\; y = 0\, and\; for\; y = \pi,\; u = 0 \quad for\; x = 0 \;\; and \;\; u = \cos^2 y = \frac{1}{2}(1+\cos 2y) \quad for\; x = \pi.$$

4. Find the harmonic function in the square $\{0 < x < a, 0 < y < 1\}$ with

Notes

the boundary conditions $u(x,0)=x, u(x,1)=0, u_x(0,y)=0, u_x(1,y)=y^2$.

4. Solve Example 1 in the case $b=1, g(x)=h(x)=k(x)=0\, but\ j(x)$ an arbitrary function.

5. Solve the following Neumann problem in the cube $\{0<x<1, 0<y<1, 0<z<1\}:\Delta u\ with\, u_z(x,y,1)=g(x,y)$ and homogeneous Neumann conditions on the other five faces, where $g(x,y)$ is an arbitrary function with zero average.

(a) Find the harmonic function in the semi-infinite strip $\{0\le x\le \pi, 0\le y\le \infty\}$ that satisfies the "boundary conditions":

$$u(0,y)=u(\pi,y)=0,\ u(x,0)=h(x), \lim_{y\to\infty} u(x,y)=0.$$

(c) What would go awry if we omitted the condition at infinity?

6.5 POISSON'S FORMULA

A much more interesting case is the Dirichlet problem for a circle. The rotational invariance of Δ provides a hint that the circle is a natural shape for harmonic functions.

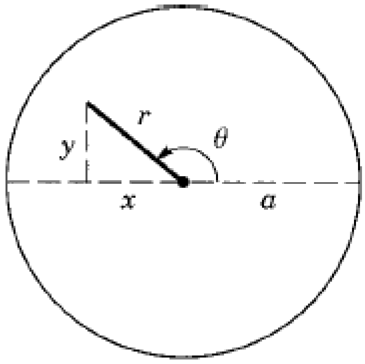
Let's consider the problem

$$\begin{aligned} u_{xx}+u_{yy}&=0 \qquad for x^2+y^2<a^2 \\ u&=h(\theta) \quad for x^2+y^2=a^2 \end{aligned}$$

With radius a and any boundary data $h(\theta)$.

Our method, naturally, is to separate variables in polar coordinates: $u=R(r)\Xi(\theta)$ (see Figure 1). From (6.1.5) we can write

$$0 = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$
$$= R'' + R' + \frac{1}{r^2}R''.$$



Dividing by R□ and multiplying by r^2, we find that

$$\Theta'' + \lambda\Theta = 0$$
$$r^2R'' + rR' - \lambda R = 0.$$

These are ordinary differential equations, easily solved. What boundary conditions do we associate with them?

For $\Theta(\theta)$ we naturally require periodic BCs:

$$\Theta(\theta + 2\pi) = \Theta(\theta) \qquad for\ -\infty < \theta < \infty.$$

Thus

$$\lambda = n^2 \quad and \quad \Theta(\theta) = A\cos n\theta + B\sin n\theta \quad (n = 1, 2, \ldots).$$

There is also the solution $\lambda = 0\, with\, \Xi(\theta) = A.$

The equation for R is also easy to solve because it is of the Euler type with solutions of the form $R(r) = r^{\alpha}$. Since $\lambda = n^2$ it reduces to

$$\alpha(\alpha - 1)r^a + \alpha r^{\alpha} - n^2 r^{\alpha} = 0$$

Whence $\alpha = \pm n.$ Thus $R(r) = Cr^n + Dr^{-n}$ and we have the separated solutions

$$u = \left(Cr^r + \frac{D}{r^n}\right)(A\cos n\theta + B\sin n\theta)$$

Notes

For $n = 1, 2, 3, \ldots.$ In case $n = 0,$ we need a second linearly independent solution of (4) (besides R =constant). It is $R = \log r,$ as one learns in ODE courses. So we also have the solutions

$$u = C + D \log r.$$

(They are the same ones we observed back at the beginning of the chapter.)

All of the solutions (8) and (9) we have sound are harmonic functions in the disk $D,$ except that half of them are infinite at the origin $(r = 0).$ But we haven't yet used any boundary condition at all in the r variable. The interval is $0 < r < a.$ At $r = 0$ some of the solutions $\left(r^{-n} \; and \log r\right)$ are infinite: We reject them. The requirement that they are finite is the "boundary condition" at $r = 0.$ Summing the remaining solutions, we have

$$u = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos n\theta + B_n \sin n\theta\right).$$

Finally, we use the inhomogeneous BCs at $r = a.$ Setting $r = a$ in the series above, we require that

$$h(\theta) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} a^n \left(A_n \cos n\theta + B_n \sin n\theta\right).$$

This is precisely the full Fourier series for $h(\theta),$ so we know that

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \cos n\phi \, d\phi$$

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \sin n\phi \, d\phi.$$

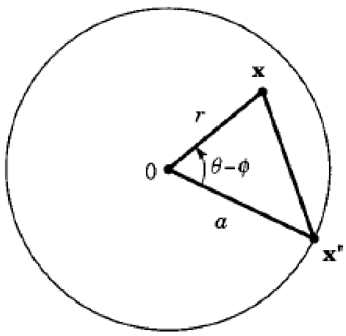
Equations (10) to(12) constitute the full solution of our problem.

Now comes an amazing fact. The series (10) can be summed explicitly! In fact, let's plug (11) and (12) directly into (10) to get

$$u(r,\theta)=\int_0^{2\pi}h(\phi)\frac{d\phi}{2\pi}$$
$$+\sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_0^{2\pi}h(\phi)\{\cos n\phi\cos n\theta+\sin n\phi\sin n\theta\}\,d\phi$$
$$=\int_0^{2\pi}h(\phi)\left\{1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\phi)\right\}\frac{d\phi}{2\pi}.$$

The term in braces is exactly the series we summed before in Section 5.5 by writing it as a geometric series of complex numbers; namely,

$$1+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)}+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)}$$
$$=1+\frac{re^{i(\theta-\phi)}}{a-re^{-i(\theta-\phi)}}+\frac{re^{-i(\theta-\phi)}}{a-re^{-i(\theta-\phi)}}$$
$$=\frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}.$$



Therefore,

$$u(r,\theta)=\left(a^2-r^2\right)\int_0^{2\pi}\frac{h(\phi)}{a^2-2ar\cos(\theta-\phi)+r^2}\frac{d\phi}{2\pi}.$$

This single formula (13), known as Poisson's formula, replaces the triple of formulas (10)-(12). It expresses any harmonic function inside a circle in terms of its boundary values.

The Poisson formula can be written in a more geometric way as follows. Write $X=(x,y)$ as a point polar coordinates (r,θ) (see Figure 2). We could also think of X as the vector from the origin 0 to the point (x,y). Let $X\,'$ be a point on the boundary.

$$X:Polar\,coordinates(r,\theta)$$
$$X\,':Polar\,coordinates(a,\phi).$$

Notes

The origin and the points $X \ and \ X'$ form a triangle with sides $r = |X|, a = |X'|, and \ |X - X'|.$ By the law of cosines

$$|X - X'|^2 = a^2 + r^2 - 2ar\cos(\theta - \phi).$$

The are length element on the circumference is $ds' = a\,d\phi.$ Therefore, Poisson's formula takes the alternative form

$$u(X) = \frac{a^2 - |X|^2}{2\pi a}\int_{|X'|=a}\frac{u(X')}{|X - X'|^2}ds'$$

for $X \in D,$ where we write $u(X') = h(\phi).$ This is a line integral with respect to arc length $ds' = a\,d\phi,$ since $s' = a\phi$ for a circle. For instance, in electrostatics this formula (14) expresses the value of the electric potential due to a given distribution of charges on a cylinder that are uniform along the length of the cylinder.

A careful mathematical statement of Poisson's formula is as follows. Its proof is given below, just prior to the exercises.

Theorem 1. Let $h(\phi) = u(X')$ be any continuous function on the circle $C = bdy\,D.$ Then the Poisson formula (13), or (14), provides the only harmonic function in D for which

$$\lim_{X \to X_0} u(X) = h(X_0) \qquad for\,all\; X_0 \in C.$$

This means that $u(X)$ is a continuous function on $\bar{D} = D \cup C.$ It is also differentiable to all orders inside D.

The Poisson formula has several important consequences. The key one is the following.

MEAN VALUE PROPERTY

Let u be a harmonic function in a disk $D,$ continuous in its closure $\bar{D}.$ Then the value of u at the centre of D equals the average of u on its circumference.

Proof. Choose coordinates with the origin 0 at the centre of the circle. Put $X = 0$ in Poisson's formula (14), or else put $r = 0$ in (13). Then

$$u(0) = \frac{a^2}{2\pi a}\int_{|X'|=a} \frac{u(X')}{a^2} ds'.$$

This is the average of u on the circumference $|X'| = a.$

MAXIMUM PRINCIPLE

This was stated and partly proved in Section 6.1. Here is a complete proof of its strong form. Let $u(X)$ be harmonic in $D.$ The maximum is attained somewhere (by the continuity of $u\, on\, \bar{D}$, say at $X_M \in \bar{D}$. We have to show that $X_M \notin \bar{D}$ unless $u \equiv$constant. By definition of M, we know that

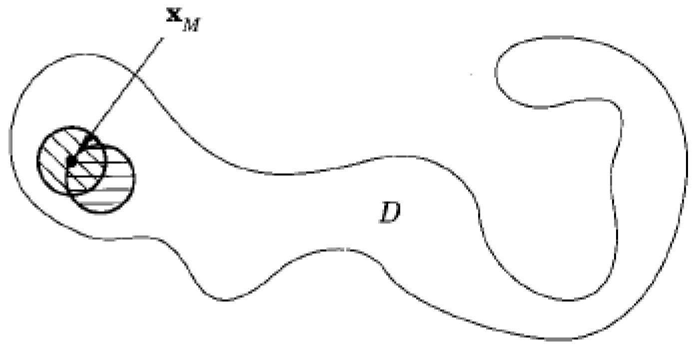
$$u(X) \le u(X_M = M) \qquad for\, all\; X \in D.$$

We draw a circle around X_M entirely contained in D (see Figure 3). By the mean value property, $u(X_M)$ is equal to its average around the circumference. Since the average is no greater than the maximum, we have the string of inequalities

$$M = u(X_M) = average\, on\, circle \le M.$$

Therefore, $u(X) = M$ for all X on the circumference. This is true for any such circle. So $u(X) = M$ for all X in the diagonally shaded region (see Figure 3). Now we repeat the argument with a different centre. We can fill the whole domain up with circles. In this way, using the assumption that D is connected, we reduce that $u(X) = M$ throughout D. So $u \equiv cons\tan t.$

Notes



DIFFERENTIABILITY

Let u be a harmonic function in any open set D of the plane. Then means that $\partial u/\partial x, \partial u/\partial y, \partial^2 u/\partial x^2, \partial^2 u/\partial x\partial y, \partial^{100} u/\partial x^{100}$, and so on, exist automatically. Let's show this first for the case where D is a disk with its centre at the origin. Look at Poisson's formula in its second form (14). The integrand is differentiable to all order for $X \in D.$ Note that $X' \in bdy\, D$ so that $X \neq X'.$ By the theorem about differentiating integrals (Section A.3), we can differentiate under the integral sign. So $u(X)$ is differentiable to any order in $D.$

Second, let D be any domain at all, and let $X_0 \in D.$ Let B be a disk contained in D with centre at $X_0.$ We just showed that $u(x)$ is differentiable inside $B,$ and hence at $X_0.$ But X_0 is an arbitrary point in $D.$ So u is differentiable (to all orders) at all points $D.$

This differentiability property is similar to the one we saw in Section 3.5 for the one-dimensional diffusion equation, but of course it is not at all true for the wave equation.

PROOF OF THE LIMIT

We begin the proof by writing (13) in the form

$$u(r,\theta) = \int_0^{2\pi} P(r,\theta-\phi)h(\phi)\frac{d\phi}{2\pi}$$

for $r < a,$ where

$$P(r,\theta) = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n\theta$$

is the Poisson kernel. Note that p has the following three properties.

$p(r,\theta) > 0 \; for\, r < a.$ This property follows from the observation that

$$a^2 - 2ar\cos\theta = r^2 \geq a^2 - 2ar + r^2 = (a-r)^2 > 0.$$

$$\int_0^{2\pi} p(r,\theta)\frac{d\theta}{2\pi} = 1.$$

This property follows from the second part of (17) because

$$\int_0^{2\pi} \cos n\theta d\theta = 0 \; for\; n = 1, 2, \ldots.$$

$p(r,\theta)$ is a harmonic function inside the circle. This property follows from the fact that each term $\left(r/a\right)^n \cos n\theta$ in the series is harmonic and therefore so is the sum.

Now we can differentiate under the integral sign (as in Appendix A.3) to get

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= \int_0^{2\pi} \left(p_{rr} + \frac{1}{r}p_r + \frac{1}{r^2}p_{\theta\theta} \right)(r,\theta-\phi)h(\phi)\frac{d\phi}{2\pi} \\ &= \int_0^{2\pi} 0.h(\phi)d\phi = 0 \end{aligned}$$

for $r < a$. So u is harmonic in $D.$

So it remains to prove (15). To do that, fix an angle θ_0 and consider a radius r near $a.$ Then we will estimate the difference

$$u(r,\theta_0) - h(\theta_0) = \int_0^{2\pi} p(r,\theta_0-\phi)\left[h(\phi) - h(\theta_0)\right]\frac{d\phi}{2\pi}$$

by property (ii) of $p.$ But $p(r,\theta)$ is concentrated near $\theta = 0.$ This is true in the precise sense that, for $\delta \leq \theta \leq 2\pi - \delta,$

$$\left|p(r,\theta)\right| = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = \frac{a^2 - r^2}{(a-r)^2 + 4ar\sin^2\left(\theta/2\right)} < \in$$

for r sufficiently close to a. Precisely, for each (small) $\delta > 0$ and each (small) $\in > 0,$ (19) is true for r sufficiently close to $a.$ Now from property (i), (18), and (19), we have

Notes

$$\left|u(r,\theta_0)-h(\theta_0)\right| \le \int_{\theta_0-\delta}^{\theta_0+\delta} p(r,\theta_0-\phi) \in \int_{|\phi-\theta_0|>\delta} \left|h(\phi)-h(\theta_0)\right| \frac{d\phi}{2\pi}$$

for r sufficiently close to a. The $\in$ in the first integral came from the continuity of h. In fact, there is some $\delta > 0$ such that $\left|h(\phi)-h(\theta)_0\right| <\in for \left|\phi-\theta_0\right| < \delta.$ Since the function $|h| \le H$ for some constant H, and in view of property (ii), we deduce from (20) that

$$\left|u(r,\theta_0)-h(\theta_0)\right| \le (1+2H)\in$$

Provided r is sufficiently close to a. This is relation (15).

EXERCISE:

Suppose that u is a harmonic function in the disk $D=\{r<2\}$ and that $u = 3\sin 2\theta + 1 \; for \; r = 2.$ Without finding the solution, answer the following questions.

Find the maximum value of $u \, in \, \bar{D}.$

Calculate the value of u at the origin.

Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with the boundary condition

$$u = 1 + 3\sin\theta \quad on \, r = a.$$

Same for the boundary condition $u = \sin^3\theta.$ (Hint: Use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$)

Show that $p(r,\theta)$ is a harmonic function in D by using polar coordinates. That is, use(6.1.5) on the first expression in (17).

Check your progress

2. Explain about Poisson's formula

6.6 CIRCLES, WEDGES, AND ANNULI

The technique of separating variables in polar coordinates works for domains whose boundaries are made up of concentric circles and rays. The purpose of this section is to present several examples of this type. In each case we get the expansion as an infinite series. (But summing the series to get a Poisson-type formula is more difficult and works only in special cases.) The geometries we treat here are

$$\begin{aligned} &A\,Wedge: \{0 < \theta < \theta_0, 0 < r < a\} \\ &An\,annulus: \{0 < a < r < b\} \\ &The\,Exterior\,of\;a\,circle: \{a < r < \infty\} \end{aligned}$$

We could do Dirichlet, Neumann, or Robin boundary conditions. This leaves us with a lot of possible examples!

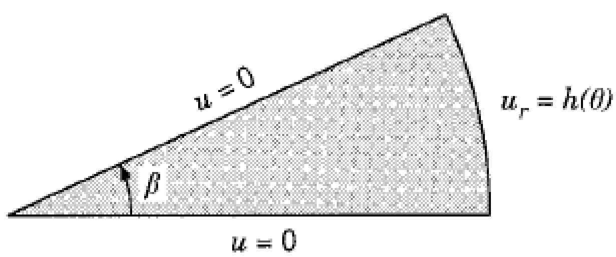
Example 1. The Wedge

Let us take the wedge with three sides $\theta = 0, \theta = \beta, \text{and}\; \mathrm{r} = \mathrm{a}$ and solve the Laplace equation with the homogeneous Dirichlet condition on the straight sides and the inhomogeneous Neumann condition on the curved side (see Figure1). That is, using the notation $u = u(r,\theta)$, the BCs are

$$u(r,0) = 0 = u(r,\beta), \qquad \frac{\partial u}{\partial r}(a,\theta) = h(\theta).$$

The separation-of-variables technique works just as for the circle, namely,

$$\Theta'' + \lambda\Theta = 0, \qquad r^2 R'' + rR' - \lambda r = 0.$$



So the homogeneous conditions lead to

$$\Theta'' + \lambda\Theta = 0, \qquad \Theta(0) = \Theta(\beta) = 0.$$

This is our standard eigenvalue problem, which has the solutions

$$\lambda = \left(\frac{n\pi}{\beta}\right)^2, \qquad \Xi(\theta) = \sin\frac{n\pi\theta}{\beta}$$

As in Section 6.3, the radial equation

$$r^2 R'' + rR' - \lambda R = 0$$

is an ODE with the solutions $R(r) = r^\alpha$, where $\alpha^2 - \lambda = 0 \, or \, \alpha = \pm\sqrt{\lambda} = \pm {n\pi}/{\beta}$. The negative exponent is reject again because we are looking for a solution $u(r,\theta)$s that is continuous in the wedge as well as its boundary: the function $r^{-n\pi/\beta}$ is infinite at the origin (which is a boundary point of the wedge). Thus we end up with the series

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/\beta} \sin\frac{n\pi\theta}{\beta}.$$

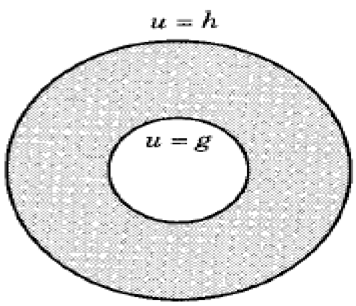
Finally, the inhomogeneous boundary condition requires that

$$h(\theta) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{\beta} \mathrm{a}^{-1+n\pi/\beta} \sin\frac{n\pi\theta}{\beta}.$$

This is just a Fourier sine series in the interval $[0,\beta]$, so its coefficients are given by the formula

$$A_n a^{1-n\pi/\beta} \frac{2}{n\pi} \int_0^\beta h(\theta) \sin\frac{n\pi\theta}{\beta} \, d\theta.$$

The complete solution is given by (5) and (6).



Example 2. The Annulus

The Dirichlet problem for an annulus (see Figure 2) is

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0 \qquad in\, 0 < a^2 < x^2 + y^2 < b^2 \\
u &= g(\theta) \qquad for\, x^2 + y^2 = a^2 \\
u &= h(\theta) \qquad for\, x^2 + y^2 = b^2
\end{aligned}
$$

The separated solutions are just the same as for a circle except that we don't throw out the functions $r^{-n}\, and \log r$, as these functions are perfectly finite within the annulus. So the solution is

$$
u(r,\theta) = \frac{1}{2}\left(C_0 + D_0 \log r\right) + \sum_{n=1}^{\infty}\left(C_n r^n + D_n r^{-n}\right)\cos n\theta
$$
$$
+\left(A_n r^n + B_n r^{-n}\right)\sin n\theta.
$$

The coefficients are determined by setting $r = a\, and\, r = b$ (see Exercise 3).

Example 3. The Exterior of a Circle

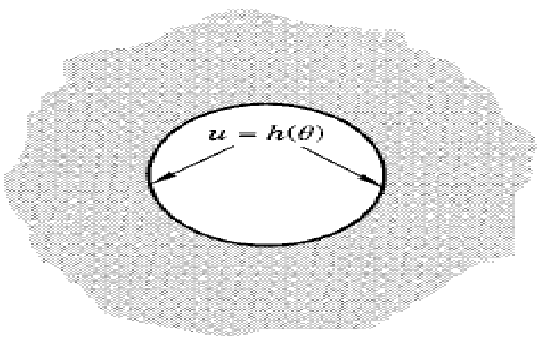
The Dirichlet problem for the exterior of a circle (see Figure 3) is

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0 \qquad for\, x^2 + y^2 > a^2 \\
u &= h(\theta) \quad for\, x^2 + y^2 = a^2 \\
u\, bounded\, as\, x^2 &+ y^2 \to \infty.
\end{aligned}
$$

We follow the same reasoning as in the interior case. But now, instead of finiteness at the origin, we have imposed boundedness at infinity.

Therefore, r^{+n} is excluded and r^{-n} is retained. So we have

$$
u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}\left(A_n \cos n\theta + B_n \sin n\theta\right).
$$



The boundary condition means

$$h(\theta)\frac{1}{2}A_0 + \sum a^{-n}\left(A_n \cos n\theta + B_n \sin n\theta\right).$$

So that

$$A_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos n\theta d\theta$$

and

$$B_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin n\theta d\theta.$$

This is the complete solution but it is one of the rare cases when the series can actually be summed. Comparing it with the interior case, we see that the only difference between the two sets of formulas is that $r \, and \, a$ are replaced by $r^{-1} \, and \, a^{-1}.$ Therefore, we get Poisson's formula with only this alteration. The result can be written as

$$u(r,\theta) = \left(r^2 - a^2\right)\int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\frac{d\phi}{2\pi}$$

for $r > a.$

These three examples illustrate the technique of separating variables in polar coordinates. A number of other examples are given in the exercises. What is the most general domain that can be treated by this method?

EXERCISE:

1. Solve $u_{xx} + u_{yy} = 0$ in the exterior $\{r > a\}$ of a disk, with the boundary condition $u = 1 + 3\sin\theta \, on \, r = a,$ and the condition at infinity that u be bounded as $r \to \infty.$

2. Solve $u_{xx} + u_{yy} = 0$ in the disk $r < a$ with the boundary condition

$$\frac{\partial u}{\partial r} - hu = f(\theta),$$

Where $f(\theta)$ is an arbitrary function. Write the answer in terms of the Fourier coefficients of $f(\theta)$.

3. (a) Find the steady-state temperature distribution inside an annular plate $\{1 < r < 2\}$, whose outer edge $(r = 2)$ is insulated, and on whose inner edge $(r = 1)$ the temperature is maintained as $\sin^2 \theta$. (Find explicitly all the coefficients, etc.)

(b)Same, except $u = 0$ on the outer edge.

4. Find the harmonic function u in the semi disk $\{r < 1, 0 < \theta < \pi\}$ with u vanishing on the diameter $\{\theta = 0, \pi\}$ and

$$u = \pi \sin \theta - \sin 2\theta \quad on\, r = 1.$$

5. Solve the problem $u_{xx} + u_{yy} = 0\, in\, D$, with $u = 0$ on the two straight sides, and $u = h(\theta)$ on the arc, where D is the wedge of Figure 1, that is, a sector of angle β cut out of a disk of radius α. Write the solution as a series, but don't attempt to sum it.

6. An annular plate with inner radius α and outer radius b is held at temperature B at its outer boundary and satisfies the boundary condition $\partial u / \partial r = A$ at inner boundary, where $A\, and\, B$ are constants. Find the temperature if it is at a steady state. (Hint: It satisfies the two-dimensional Laplace equation and depends only on r.)

7. Solve $u_{xx} + u_{yy} = 0$ in the quarter-disk $\{x^2 + y^2 < a^2, x > 0, y > 0\}$ with the following BCs:

$$u = 0 \quad on\, x = 0\, and\, on\, y = 0 \; and \; \frac{\partial u}{\partial r} = 1\, on\, r = a.$$

Write the answer as an infinite series and write the first two nonzero terms explicitly.

8. Prove the uniqueness of the Robin problem

$$\Delta u = f \quad in\, D, \quad \frac{\partial u}{\partial n} + au = h \; on\, bdy\, D,$$

Where D is any domain in three dimensions and where α is a positive constant.

6.7 LET US SUM UP

In this unit we have discussed about Laplace equation, A solution of the Laplace equation is called a harmonic function. From Maxwell's equations, one has curl $E=0$ and div $E=4\pi p$, where p is the charge density.

Maximum Principle: Let D be a connected bounded open set (in either two- or three-dimensional space). Let either $u(x,y) or\, u(x,y,z)$ be a harmonic function in D that is continuous on $\bar{D}=D\cup(bdy\,D)$. Then the maximum and minimum values of u are attained on $bdy\,D$ and nowhere inside $(unless\, u\equiv cons\tan t)$.

Mean value property: Let u be a harmonic function in a disk D, continuous in its closure $\bar{D}$. Then the value of u at the centre of D equals the average of u on its circumference.

6.8 KEY WORDS

1. Both the diffusion and the wave equations reduce to the Laplace equation.
2. A solution of the Laplace equation is called a harmonic function.
3. From Maxwell's equations, one has curl $E=0$ and div $E=4\pi p$, where p is the charge density.
4. The maximum principle, which is easier for Laplace's equation than for the diffusion equation.
5. The rotational invariance suggests that the two-dimensional laplacian

$$\Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

6. The technique of separating variables in polar coordinates works for domains whose boundaries are made up of concentric circles

and rays. The purpose of this section is to present several examples of this type.

6.9 QUESTIONS FOR REVIEW

1. Discuss about Laplace's equation
2. Discuss about Rectangles and cubes
3. Discuss about Poisson's formula
4. Discuss about Circles, Wedges and annuli

6.10 SUGGESTED READINGS AND REFERENCES

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6.11 ANSWERS TO CHECK YOUR PROGRESS

1. See section 6.2

2. See section 6.5

UNIT-7 GREEN’S IDENTITIES AND GREEN’S FUNCTIONS

STRUTURE

7.0 OBJECTIVE

In this unit we will learn ad discuss about Green’s first identity, Green’s second identity, Green’s functions and Half space and sphere.

7.1 INTRODUCTION

The Green’s identities for the laplacian lead directly to the maximum principle and to Dirichlet’s principle about minimizing the energy.

The Green’s function is a kind of universal solution for harmonic functions in a domain. All other harmonic functions can be expressed in terms of it.

Combined with the method of reflection, the Green's function leads in a very direct way to the solution of boundary problems in special geometries.

George Green was interested in the new phenomena of electricity and magnetism in the early 19th century.

7.2 GREEN'S FIRST IDENTITY

NOTATION

In this chapter the divergence theorem and vector notation will be used extensively. Recall the notation (in three dimensions)

$$grad\ f = \nabla f = the\, vector\left(f_x, f_y, f_z\right)$$
$$div\, F = \nabla.F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

Where $F = \left(F_1, F_2, F_3\right)$ is a vector field. Also,

$$\Delta u = div\, grad\, u = \nabla.\nabla u = u_{xx} + u_{yy} + u_{zz}$$
$$\left|\nabla u\right|^2 = \left|grad\, u\right|^2 = u_x^2 + u_y^2 + u_z^2.$$

Watch out which way you draw the triangle: in physics texts one often finds the laplacian $\nabla.\nabla$ written as ∇^2, but write it as ∇.

We will write almost everything in this chapter for the three-dimensional case. (However, using two dimensions is okay, too, even n dimensions.) Thus we write

$$\iiint_D \ldots dX = \iiint_D \ldots \mathrm{dx\, dy\, dz}$$

if D is a three-dimensional region (a solid), and

$$\int\int_{bdy\, D} \ldots dS = \iint_s \ldots dS,$$

Where $S = bdyD$ is the bounding surface for the solid region D. Here dS indicates the usual surface integral, as in the calculus.

Our basic tool in this chapter will be the divergence theorem:

$$\iiint_D divF\, dX = \iint_{bdy\,D} F.ndS,$$

Where F is any vector function, D is a bounded solid region, and n is the unit outer normal on $bdy\,D$ (see Figure 1) (see Section A.3).

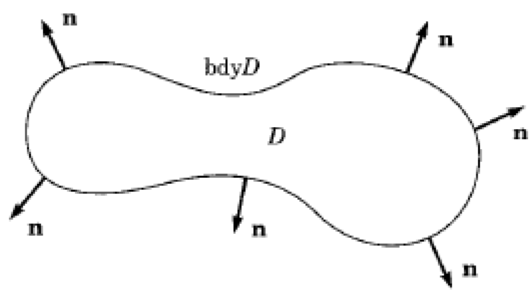
GREEN'S FIRST IDENTITY

We start from the product rule

$$(\upsilon u_x)_x = \upsilon_x u_x + \upsilon u_{xx}$$

and the same with $y\, and\, z$ derivatives. Summing, this leads to the identity

$$\Delta.(\upsilon\nabla u) = \nabla\upsilon.\nabla u + \upsilon\Delta u.$$



These we integrate and use the divergence theorem on the left side to get

$$\iint_{bdy\,D} \upsilon \frac{\partial u}{\partial n} dS = \iiint_D \nabla\upsilon.\nabla u dX + \iiint_D \upsilon\Delta u\, dX,$$

Where $\partial u/\partial n = n.\nabla u$ is the directional derivative in the outward normal direction. This is Green's first identity. It is valid for any solid region D and any pair of functions $u\, and\, \upsilon.$ For example, we could take $\upsilon \equiv 1$ to get

$$\iint_{bdy\,D} \frac{\partial u}{\partial n} d\,S = \iiint_D \Delta u\, dX.$$

As an immediate application of (2), consider the Neumann problem in any domain $D.$ That is,

$$\begin{cases} \Delta u = f(x) & in\, D \\ \dfrac{\partial u}{\partial n} = h(x) & on\, bdy\, D. \end{cases}$$

By (2) we have

$$\int\int_{bdy\, D} h\, d\, S = \iiint_D f\, dX.$$

It follows that the data $(f\ and\ h)$ are not arbitrary but are required to satisfy condition (4). Otherwise, there is no solution. In that sense the Neumann problem (3) is not completely well-posed. On the other hand, one can show that if (4) is satisfied, then (3) does have a solution so the situation is not too bad.

What about uniqueness in problem (3)? Well, you could add any constant to any solution of (3) and still get a solution. So problem (3) lacks uniqueness as well as existence.

MEAN VALUE PROPERTY

In three dimensions the mean value property states that the average value of any harmonic function over any sphere equals its value at the center. To prove this statement, let D is the sphere (surface) $\{|X| < a\}$, say; that is, $\{x^2 + y^2 + z^2 < a^2\}$. then $bdy\, D$ is the sphere (surface) $\{|X| = a\}$. Let $\Delta u = 0$ in any region that contains D and $bdy\, D$. For a sphere, n points directly away from the origin, so that

$$\frac{\partial u}{\partial n} = n.\Delta u = \frac{X}{r}.\Delta u = \frac{x}{r}u_x + \frac{y}{r}u_y + \frac{z}{r}u_z = \frac{\partial u}{\partial r},$$

Where $r = \left(x^2 + y^2 + z^2\right)^{1/2} = |X|$ is the spherical coordinate, the distance of the point (x, y, z) from the centre 0 of the sphere. Therefore, (2) becomes

$$\int\int_{bdy\, D} d\, S = 0.$$

Notes

Let's write this integral in spherical coordinates, (r,θ,ϕ). Explicitly, (5) takes the form

$$\int_0^{2\pi}\int_0^{\pi} u_r\left(a,\theta,\phi\right)a^2\sin\theta d\theta d\phi = 0$$

Since $r = a\, on\, bdy\, D.$ We divide this by the constant $4\pi a^2$ (the area of $bdy\, D$). This result is valid for all $a > 0,$ so that we can think of a as a variable and call it $r.$ Then we pull $\partial/\partial r$ outside the integral (see Section A.3), obtaining

$$\frac{\partial}{\partial r}\left[\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u\left(r,\theta,\phi\right)\sin\theta d\theta d\phi\right] = 0.$$

Thus

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u\left(r,\theta,f\right)\sin\theta d\theta d\phi$$

is independent of $r.$ This expression is precisely the average value of u on the sphere $\{|X| = r\}.$ In particular, if we let $r \to 0,$ we get

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u\left(o\right)\sin\theta d\theta d\phi = u\left(0\right).$$

That is,

$$\frac{1}{area of\ S}\int\int_s u\, dS = u\left(0\right).$$

This proves the mean value property in three dimensions.

Actually, the same idea works in n dimensions. For $n = 2$ recall that we found another proof in Section 6.3 by a completely different method.

MAXIMUM PRINCIPLE

Exactly as in two dimensions in Section 6.3, we deduce from the mean value property the maximum principle.

If D is any solid region, aa non constant harmonic function in D cannot take its maximum value inside D, but only on $bdy\, D.$

It can also be shown that the outward normal derivative $\partial u/\partial n > 0$ there. The last assertion is called the Hopf maximum principle. For a proof, see $[PW]$.

UNIQIQUENESS OF DIRICHLET'S PROBLEM

We gave one proof in Section 6.1 using the maximum principle. Now we give another proof by the energy method. If we have two harmonic functions $u_1 \, and \, u_2$ with the same boundary data, then their difference $u = u_1 - u_2$ is harmonic and has zero boundary data. We go back to (GI) and substitute $\upsilon = u$. Since u. Since u is harmonic, we have $\Delta u = 0$ and

$$\int_{bdy\,D}\int u \frac{\partial u}{\partial n} d\,S = \iiint_D |\nabla u|^2 \, dX.$$

Since $u = 0 \, on \, bdy \, D$, the left side of (7) vanishes. Therefore, $\iiint_D |\nabla u|^2 \, dX = 0$. By the first vanishing theorem in Section A.1, it follows that $|\nabla u|^2 \equiv 0 \, in \, D$. Now a function with vanishing gradient must be a constant (provided that D is connected). So $u(X) \equiv C$ throughout D. But u vanishes somewhere (on $bdy\,D$), so C must be 0. Thus $u(X) \equiv 0 \, in \, D$. This proves the uniqueness of the Dirichlet problem.

Uniqueness of Neumann's problem: If $\Delta u = 0 \, in \, D \, and \, \partial u/\partial n = 0 \, on \, bdy \, D$, then u is a constant in D .

DIRICHLET'S PRINCIPLE

This is an important mathematical theorem based on the physical idea of energy. It states among all the functions $w(X)$ in D that satisfy the Dirichlet boundary condition

$w = h(X) \quad on \, bdy \, D,$

the lowest energy occurs for the harmonic function satisfying (8). In the present Context the energy is defined as

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2 \, dX.$$

This is the pure potential energy, there being no kinetic energy because there is no motion. Now it is a general principle in physics that any system prefers to go to the state of lowest energy, called the ground state. Thus the harmonic function is the preferred physical stationary state. Mathematically, Dirichlet's principle can be stated precisely as follows:

Let $u(X)$ be the unique harmonic function in D that satisfies (8). Let $w(X)$ be any function in D that satisfies (8). Then

$$E[w] \geq E[u].$$

To prove Dirichlet's principle, we let $\upsilon = u - w$ and expand the square in the integral

$$E[w] = \frac{1}{2}\iiint_D |\nabla(u-\upsilon)|^2 \, dX$$
$$= E[u] - \iiint_D \nabla u . \nabla \upsilon dX + E[\upsilon].$$

Next we apply Green's first identity (GI) to the pair functions $u \, and \, \upsilon.$ In (GI) two of the three are zero because $\upsilon = 0$ on $bdy \, D \, and \, \Delta u = 0 \, in \, D.$ Therefore, the middle term in (11) is also zero. Thus

$$E[w] = E[u] + E[\upsilon].$$

Since it is obvious that $E[\upsilon] \geq 0,$ we deduce that $E[w] \geq E[u].$ This means that the energy is smallest when $w = u.$ This proves Dirichlet's principle.

An alternative proof goes as follows. Let $u(X)$ be a function that satisfies (8) and minimizes the energy (9). Let $\upsilon(X)$ be any function that vanishes on $bdy\,D.$ Then $u+\in\upsilon$ satisfies the boundary condition (8). So if the energy is smallest for the function $u,$ we have

$$E[u]\leq E[u+\in\upsilon]=E[u]-\in\iiint_D \Delta u\upsilon dX+\in^2 E[\upsilon]$$

For any constant $\in$. The minimum occurs for $\in=0.$ By calculus,

$$\iiint_D \Delta u\upsilon dX=0.$$

This is valid for practically all functions $\upsilon\, in\, D.$ Let D' be any strict sub domain of $D;$ that is, $\bar{D}'\subset D.$ Let $\upsilon(X)\equiv 1\; for\; X\in D'\, and\, \upsilon(X)\equiv 0\; for\; X\in D-D'.$ In (13) we choose this function $\upsilon.$ (Because this υ is not smooth, an approximation argument is required that is omitted here.) Then (13) takes form

$$\iiint_{D'} \Delta u dX=0 \qquad for\, all\, D'.$$

By the second vanishing theorem in Section A.1, it follows that $\Delta u=0$ in $D.$ Thus $u(X)$ is a harmonic function. By uniqueness, it is the only function satisfying (8) that can minimize the energy.

EXERCISE:

1. Derive the three-dimensional maximum principle from the mean value property.
2. Prove the uniqueness up to constants of the Neumann problem using the energy method.
3. Prove the uniqueness of the Robin problem $\partial u/\partial n+a(X)u(X)=h(X)$ provided that $a(X)>0$ on the boundary.
4. Generalize the energy method to prove uniqueness for the diffusion equation with Dirichlet boundary conditions in three conditions.
5. Prove Dirichlet's principle for the Neumann boundary condition. It asserts that among all real-valued functions $w(X)\, on\, D$ the quantity

Notes

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2\, dX - \iint_{bdy\,D} hw\, dS$$

is the smallest for $w-u,$ where u is the solution of the Neumann problem

$$-\Delta u = 0\, in\, D, \quad \frac{\partial u}{\partial n} = h(X)\, on\, bdy\, D.$$

It is required to assume that the average of the given function $h(X)$ is zero (by Exercise 6.1.11).

Notice three features of this principle:

(i)There is no constraint at all on the trial functions $w(X).$

(ii) The function $h(X)$ appears in the energy.

(iii) The functional $E[w]$ does not change if a constant is added to $w(X).$

(Hint: Follow the method in Section 7.1.)

Let $A\, and\, B$ ne two disjoint bounded spatial domains, and let D be their exterior. So $bdy\, D = bdy\, A \cup bdy\, B.$ Consider a harmonic function $u(X)$ in D that tends to zero at infinity, which is constant on $bdy\, A$ and constant on $bdy\, B,$ and which satisfies

$$\iint_{bdy\,A} \frac{\partial u}{\partial n} dS = Q > 0 \quad and \quad \iint_{bdy\,B} \frac{\partial u}{\partial n} d\, S = 0.$$

[Interpretation: The harmonic function $u(X)$ is the electrostatic potential of two conductors, $A\, and\, B; Q$ is the charge on $A, while\, B$ is uncharged.]

7. Show that the solution is unique. (Hint: Use the Hopf maximum principle.)

Show that $u \geq 0\, in\, D.$ [Hint: If not, then $u(X)$ has a negative minimum. Use the Hopf principle again.]

8. Show that $u > 0 \, in \, D.$

(Rayleigh-Ritz approximation to the harmonic function $u \, in \, D$ with $u = h \, on \, bdy \, D.$) Let $w_0, w_1, ..., w_n$ be arbitrary functions such that $w_0 = \text{h}$ on bdy D and $w_1 = ... = \text{w}_n = 0 w_n$ on bdy D. The problem is to find constants $c_1, ..., c_n$ so that

$w_0 + c_1 w_1 + ... + c_n w_n$ has the least possible energy.

9. Show that the constants must solve the linear system

$$\sum_{k=1}^{n} \left(\nabla w_j, \nabla w_k \right) c_k = -\left(\nabla w_0, \nabla w_j \right) \; for \; j = 1, 2, ..., n.$$

10. Consider the problem $u_{xx} + u_{yy} = 0$ in the triangle $\{ x > 0, y > 0, 3x + y < 3 \}$ with the boundary conditions

$$u(x,0) = 0 \;\; u(0,y) = y(3-y) \;\; u(x,3-3x) = 0$$

Check your progress

1. Explain about Green’s first identity

7.3 GREEN’S SECOND IDENTITY

Green’s second identity is the higher-dimensional version of the identity. It leads to a basic representation formula for harmonic functions that we require in the next section.

The middle term in (GI) does not change if $u \, and \, \upsilon$ are switched. So of we write (GI) for the pair $u \, and \, \upsilon$, and again for the pair $u \, and \, \upsilon$, and then subtract , we get

$$\iiint_D (u \Delta \upsilon - \upsilon \Delta u) \, dX = \iint_{bdy \, D} \left(u \frac{\partial u}{\partial n} - \upsilon \frac{\partial u}{\partial n} \right) dS.$$

This is green's second identity. Just (GI), it is valid for any pair of functions $u\, and\, \upsilon.$

It leads to the following natural definition. A boundary condition is called symmetric for the operator Δ if the right side (G2) vanishes for all pairs of functions u, υ that satisfy the boundary condition. Each of the three classical boundary conditions (Dirichlet, Neumann, and Robin) is symmetric.

REPRESENTATION FORMULA

This formula represents any harmonic function as an integral over the boundary. It states the following: If $\Delta u = 0\, in\, D,$ then

$$u(X_0) = \int\int_{bdy\, D}\left[-u(X)\frac{\partial}{\partial n}\left(\frac{1}{|X - X_0|}\right) + \frac{1}{|X - X_0|}\frac{\partial u}{\partial n}\right]\frac{dS}{4\pi} \backslash$$

What is involved here is the same fundamental radial solution r^{-1} that we found in Section 6.1, but translated by the vector $X_0.$

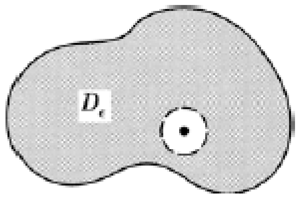
Proof of (1). The representation formula (1) is the special case of (G2) with the choice $\upsilon(X) = \left(-4\pi|X - X_0|\right)^{-1}.$ Clearly, the right side of (G2) agrees with (1). Also, $\Delta u = 0\, and\, \Delta\upsilon = 0,$ which kills the left side of (G2). So where does the left side of (1) come from? From the fact that the function $\upsilon(X)$ is infinite at the point $X_0.$

Therefore, it is forbidden to apply (G2) in the whole domain $\mathrm{D}.$ So let's take a pair of scissors and cut out a small ball around $X_0.$ Let $\mathrm{D}_\in$ be the region D with this ball (of radius $\in$ and centre X_0) excised (see Figure 1). For simplicity let X_0 be the origin. Then $\upsilon(X) = -1/(4\pi r),$ where $r = \left(x^2 + y^2 + z^2\right)^{1/2} = |X|.$ Writing down (G2) with this choice of $\upsilon,$ we have, since $\Delta u = 0 = \Delta\upsilon\, in\, D_\in,$

$$-\int\int_{bdy\, D_\in}\left[u.\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{\partial u}{\partial n}.\frac{1}{r}\right]dS = 0.$$

But $bdy\,D_\in$ consists of two parts: the original boundary $bdy\,D$ and the sphere $\{r=\in\}$. On the sphere, $\partial/\partial n = -\partial/\partial r$. Thus the surface integral breaks into two pieces.

$$-\int\int_{bdy\,D_\in}\left[u.\frac{\partial}{\partial n}\left(\frac{1}{r}\right)-\frac{\partial u}{\partial n}.\frac{1}{r}\right]dS = 0.-\int\int_{r=\in}\left[u.\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\frac{\partial u}{\partial r}.\frac{1}{r}\right]dS..$$



This identity (2) is valid for any small $\in>0$. Our representation formula (1) would follow provided that we could show that the right side of (2) tended to $4\pi u(0)\, as \in\to 0.$

Now, on the little spherical surface $\{r=\in\}$, we have

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\right)=-\frac{1}{r^2}=-\frac{1}{\in^2},$$

So that the right side of (2) equals

$$\frac{1}{\in^2}\int\int_{r=\in} u\,dS+\frac{1}{\in}\int\int_{r=\in}\frac{\partial u}{\partial r}\,d\,S = 4\pi\bar{u}+4\pi\in\frac{\overline{\partial u}}{\partial r},$$

Where $\bar{u}$ denotes the average value of $u(X)$ on the sphere $|X|=r=\in$, and $\overline{\partial u/\partial r}$ denotes the average value of $\partial u/\partial n$ on this sphere. As $\in\to 0,$ the expression (3) approaches

$$4\pi u(0)+4\pi.0.\frac{\partial u}{\partial r}(0)=4\pi u(0)$$

because u is continuous and $\partial u/\partial r$ is bounded. Thus (2) turns into (1), and this completes the proof.

The corresponding formula in two dimensions is

$$u(X_0)=\frac{1}{2\pi}\int_{bdy\,D}\left[u(X)\frac{\partial}{\partial n}\left(\log|X-X_0|\right)-\frac{\partial u}{\partial n}\log|X-X_0|\right]ds$$

Whenever $\Delta u = 0$ in a plane domain D and X_0 is a point with in D. The right side is a line integral over the boundary curve with respect to arc length. Log denotes the natural logarithm and ds the arc length on the bounding curve.

EXERCISE:

Derive the representation formula for harmonic functions (7.2.5) in two dimensions.

Let $\phi(X)$ be any C^2 function defined on all of three-dimensional space that vanishes outside some sphere. Show that

$$\phi(0) = -\iiint \frac{1}{|X|} \Delta\phi(X) \frac{dX}{4\pi}.$$

The integration is taken over the region where $\phi(X)$ is not zero.

Give yet another derivation of the mean value property in three dimensions by choosing D to be a ball and X_0 its centre in the representation formula (1).

Check your progress

2. Explain about Green's second identity

7.4 GREEN'S FUNCTIONS

We now use Green's identities to study the Dirichlet problem. The representation formula (7.2.1) used exactly two properties of the function $\upsilon(X) = \left(-4\pi|X - X_0|\right)^{-1}$: that it is harmonic except at X_0 and that it has a certain singularity there. Our goal is to modify this function so that one of the terms in (7.2.1) disappears. The modified function is called the Green's function for D.

Definition. The Green's function $G(X)$ for the operator $-\Delta$ and the domain D at the point $X_0 \in D$ is a function defined for $X \in D$ such that:

(i) $G(X)$ possesses continuous second derivatives and $\Delta G = 0 \, in \, D,$ except at the point $X = X_0$.

(ii) $G(X) = 0 \; for \; x \in bdyD.$

(iii)This function $G(X) + 1/(4\pi|X - X_0|)$ is finite at X_0 and has continuous second derivatives everywhere and is harmonic at X_0.

It can be shown that a Green's function exists. Also, it is unique by Exercise 1. The usual notation for the Green's function is $G(X, X_0)$.

Theorem 1. If $G(X, X_0)$ is the Green's function, then the solution of the Dirichlet problem is given by the formula

$$u(X_0) = \int \int_{bdy\,D} u(X) \frac{\partial G(X, X_0)}{\partial n} d\,S.$$

Proof. Let us go back to the representation formula (7.2.1):

$$u(X_0) = \int \int_{bdy\,D} \left(u \frac{\partial \upsilon}{\partial n} - \frac{\partial u}{\partial n} \upsilon \right) d\,S,$$

Where $\upsilon(X) = -(4\pi|X - X_0|)^{-1}$, as before. Now let's write $G(X, X_0) = \upsilon(X) + H(X)$. [This is the definition of $H(X)$.] Then $H(X)$ is a harmonic function throughout the domain D [by (iii) and (i)]. We apply Green's second identity (G2) to the pair of harmonic functions $u(X)$ and $H(X)$:

$$0 = \int \int_{bdy\,D} \left(u \frac{\partial H}{\partial n} - \frac{\partial u}{\partial n} h \right) dS.$$

Adding (2) and (3), we get

$$u\left(X_0\right)=\int\int_{bdy\,D}\left(u\frac{\partial G}{\partial n}-\frac{\partial u}{\partial n}G\right)dS.$$

But by (ii), G vanishes on $bdyD$, so the last term vanishes and we end up with formula (1).

The only thing wrong with this beautiful formula is that it is not usually easy to find G explicitly. Nevertheless, in the next section we'll see how to use the reflection method to find G in some situations and thereby solve the Dirichlet problem for some special geometries.

SYMMETRY OF THE GREEN'S FUNCTION

For any region D we have a Green's function $G\left(X,X_0\right)$. It is always symmetric:

$$G\left(X,X_0\right)=G\left(X_0,X\right)\qquad for\ X\neq X_0.$$

In order to prove (4), we apply Green's second identity (G2) to the pair of functions $u\left(X\right)=G\left(X,a\right)and\,\upsilon\left(X\right)=G\left(X,b\right)$ and to the domain $D_\in$. By $D_\in$ we donate the domain D with two little spheres of radii $\in$ cut out around the points $a\,and\,b$ (see Figure 1). So the boundary of $D_\in$ consists of three parts: the original boundary $bdy\,D$ and the two spheres $\left|X-a\right|=\in and\left|X-b\right|=\in$. Thus

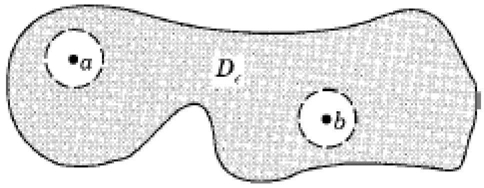
$$\iint\int_{D_\in}\left(u\Delta\upsilon-\upsilon\Delta u\right)dX=\int\int_{bdy\,D}\left(u\frac{\partial\upsilon}{\partial n}-\upsilon\frac{\partial u}{\partial n}\right)dS+A_\in+B_\in,$$

Where

$$A_\in=\int\int_{\left|X-a\right|=\in}\left(u\frac{\partial\upsilon}{\partial n}-\upsilon\frac{\partial u}{\partial n}\right)dS$$

and $B_\in$ is given by the same formula at b. Because both $u\,and\,\upsilon$ vanish on $bdy\,D$, the integral over $bdy\,D$ also vanishes. Therefore,

$A_\in+B_\in=0$ for each $\in$.



Let's calculate the limits as $\in \to 0$. We shall then have $\lim A_\in + \lim B_\in = 0$. For $A_\in$, denote $r = |X - a|$. Then

$$\lim_{\in \to 0} A_\in = \lim_{\in \to 0} \int \int_{r=\in} \left\{ \left(-\frac{1}{4\pi r} + H \right) \frac{\partial \upsilon}{\partial n} - \upsilon \frac{\partial}{\partial n} \left(-\frac{1}{4\pi r} + H \right) \right\} r^2 \sin\theta \, d\theta \, d\phi$$

Where θ and ϕ are the spherical angles for $x - a$, and H is a continuous function. Now $\frac{\partial}{\partial} = -\frac{\partial}{\partial r}$ for the sphere. Among the four terms in the last integrand. Only the third one contributes a nonzero expression to the limit (for the same reason as in the derivation of (7.2.1). Thus

$$\lim_{\in \to 0} A_\in = \lim_{\in \to 0} \int_0^{2\pi} \int_0^{\pi} \upsilon \frac{1}{4\pi \in^2} \sin\theta \, d\theta \, d\phi = \upsilon(a)$$

by cancellation of the $\in^2$, A quite similar calculation shows that $\lim B_\in = -u(b)$. therefore,

$$0 = \lim(A_\in + B_\in) = \upsilon(a) - u(b) = G(a,b) - G(b,a).$$

This proves the symmetry (4).

In electrostatics, $G(X, X_0)$ is interpreted as the electric potential inside a conducting surface $S = bdy\, D$ due to a charge at a single point X_0. The symmetry (4) is known as the principle of reciprocity. It asserts that a source located at the point a produces at the point b the same effect as a source at b would produce at a.

The Green's function also allows us to solve Poisson's equation.

Theorem 2. The solution of the problem

$$\Delta u = f \quad in\, D \qquad u = h \qquad on\, bdy\, D$$

Is given by

$$u(X_0)=\int\int_{bdy\,D} h(X)\frac{\partial G(X,X_0)}{\partial n}dS+\iiint_D f(X)G(X,X_0)dX.$$

The proof is left as an exercise.

EXERCISE:

1. Show that the Green's function is unique. (Hint: Take the difference of two of them.)
2. Prove Theorem 2, which gives the solution of Poisson's equation in terms of the Green's function.
3. Verify the limit of $A_\in$ as claimed in the proof of the symmetry of the Green's function.

7.5 HALF-SPACE AND SPHERE

We solve the harmonic functions in a half-space and a sphere by combing the Green's function with the method of reflection.

THE HALF-SURFACE

We first determine the Green's function for a half-space. A half-space is the region lying on one side of a plane. Although it is an infinite domain, all the ideas involving Green's functions are still if we impose the "boundary condition at infinity" that the functions and their derivatives tend to 0 as $|X|\to\infty$.

We write the coordinates as $X=(x,y,z)$. Say that the half-space is $D=\{z>0\}$, the domain that lies above the xy plane (see Figure 1). Each point $X=(x,y,z)$ in D has a reflected point $X^*=(x,y,-z)$ that is not in D.

Now we already know that the function $1/\left(4\pi|X-X_0|\right)$ satisfies two of the three conditions – (i) and (iii) – requires of the Green's function: We want the modify it to get (ii) as well.

We assert that the Green's function for a D is

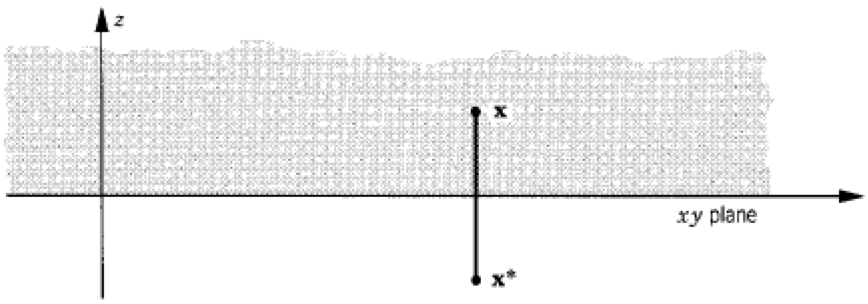
$$G(X, X_0) = -\frac{1}{4\pi|X - X_0|} + \frac{1}{4\pi|X - X_0^*|}.$$

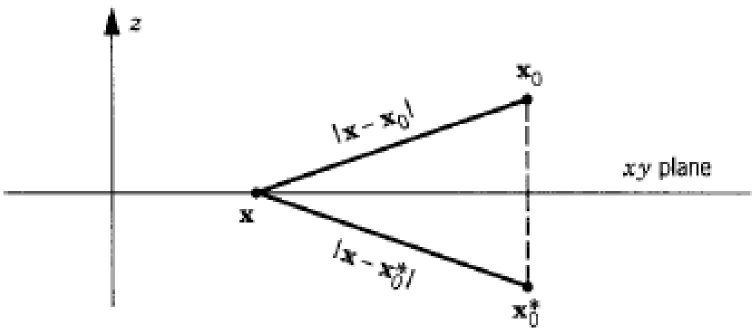
In coordinates,

$$G(X, \mathrm{x}_0) = \frac{1}{4\pi}\left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right]^{-1/2}$$
$$+ \frac{1}{4\pi}\left[(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2\right]^{-1/2}.$$

Notice that the two terms differ only in the $(z \pm z_0)$ factors. Let's verify the assertion (1) by checking each of the three properties of G.

(i) Clearly, G is finite and differentiable except at X_0. Also, $\Delta G = 0$.





(ii) This is the main property to check. Let $X \in bdy\, D$, so that $z = 0$. From Figure 2 we see that $|X - X_0| = |X - X_0^*|$. Thus $G(X, X_0) = 0$.

(iii) Because X_0^* is outside out domain D, the function $-1/\left(4\pi|X - X_0^*|\right)$ has no singularity inside the domain, so that G has the proper singularity at X_0.

(iv) These three properties prove that $G(X, X_0)$ is the Green's function for this domain. Let's now it to solve the Dirichlet problem

$$\Delta u = 0 \quad for\, z > 0, \quad u(x, y, 0) = h(x, y).$$

We use formula (7.3.1). Notice that $\partial G/\partial \mathrm{n} = -\partial G/\partial \mathrm{z}|_{z=0}$ because n points downward (outward from the domain). Furthermore,

$$-\frac{\partial G}{\partial z} = \frac{1}{4\pi}\left(\frac{z + z_0}{|X - X_0^*|^3} - \frac{z - z_0}{|X - X_0|^3}\right)$$
$$= \frac{1}{2\pi}\frac{z_0}{|X - X_0|^3}$$

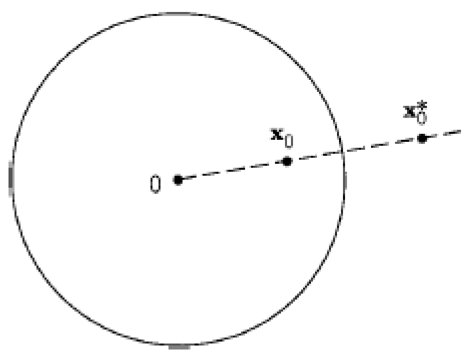
on $z = 0$. Therefore, the solution of (2) is

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi}\iint_D \left[(x - x_0)^2 + (y - y_0)^2 + (z_0)^2\right]^{-3/2} h(x, y)\, dx\, dy,$$

Where both integrals run over $(-\infty, \infty)$, noting that $z = 0$ in the integrand.

$$u(X_0) = \frac{z_0}{2\pi}\iint_D \frac{h(X)}{|X - X_0|^3}\, d\, S.$$

This is the complete formula that solves the Dirichlet problem for the half-space.



THE SPHERE

The Green's function for the ball $D = \{|X| < a\}$ of radius a can also be found by the reflection method. In this case, however, the reflection is across the sphere $\{|X| = a\}$, which is the boundary D (see Figure 3).

Fix any nonzero point X_0 in the ball $(that\, is, 0 < |X_0| < a)$. the reflected point X_0^* is defined by two properties. It is collinear with the origin 0 and the point X_0. Its distance from the origin is determined by the formula $|X_0||X_0^*| = a^2$. Thus

$$X_0^* = \frac{a^2 X_0}{|X_0|^2}.$$

If X is any point at all, let's denote $|X - X_0| = p \, and \, |X - X_0^*| = p^*$. Then the Green's function of the ball is

$$G(X, X_0) = -\frac{1}{4\pi p} + \frac{a}{|X_0|} \frac{1}{4\pi p^*}$$

if $X_0 \neq 0$. To verify formula, we need only check the three conditions (i),(ii), and (iii). We'll consider the case $X_0 = 0$ separately.

First of all, G has no singularity except at $X = X_0$ because X_0^* lies outside the ball. The functions $1/p \, and \, 1/\mathrm{p}^*$ are harmonic in D except at X_0 because they are just translates of $1/r$. Therefore, (i) and (iii) are true.

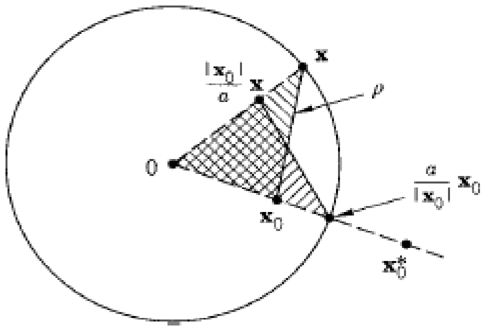
To prove (ii), we show that p^* is proportional to p for all points X on the spherical surface $|X| = a$. to do this, we notice from the congrument triangles in Figure 4 that

$$\left| \frac{r_0}{a} X - \frac{a}{r_0} x_0 \right| = |X - X_0|,$$

Where $r_0 = |X_0|$. The left side of (7) equals

$$\frac{r_0}{a} \left| X - \frac{a}{r_0^2} x_0 \right| = \frac{r_0}{a} p^*$$

Notes



Thus

$$\frac{r_0}{a}p^* = p \qquad for\ all\,|X| = a.$$

Therefore, the function

$$-\frac{1}{4\pi p} + \frac{a}{|X_0|}\frac{1}{4\pi p^*},$$

defined above is zero on the sphere $|X| = a.$ This is condition (ii). This proves formula (6).

We can also write (6) in the form

$$G(X, X_0) = \frac{1}{4\pi|X - X_0|} + \frac{1}{4\pi|r_0 X\ /\ a - aX_0\ /\ r_0|}.$$

In case $X_0 = 0,$ the formula for the Green's function is

$$G(X, 0) = -\frac{1}{4\pi|X|} + \frac{1}{4\pi a}$$

Let's now use 96) to write the formula for the solution of the Dirichlet problem in a ball:

$$\Delta u = 0 \qquad in\,|X| < a, \qquad u = h \quad on\,|X| = a.$$

We already know from Chapter 6 that $u(0)$ is the average of $h(x)$ on the sphere, so let's consider $X_0 \neq 0.$ To apply (7.3.1), we need to calculate $\partial G / \partial \mathrm{n}$ on $|X| = a$. (Let's not forget X_0 is considered to be fixed, and the derivatives are with respect to X.)

We not that $p^2 = |X - X_0|^2$. Differentiating, we have $2\rho\nabla\rho = 2(X - X_0)$. So $\nabla\rho = (X - X_0)/\rho$ and $\nabla(\rho^*) = (X - X_0^*)/\rho^*$. Hence differentiating (6), we have

$$\nabla G = \frac{X - X_0}{4\pi p^3} - \frac{aX - X_0^*}{r_0 4\pi\rho^{*3}}.$$

Remember that $\mathrm{X}_0^* = (a/r_0)^2 X_0$. If $|X| = a$, we showed above that $\rho^* = (a/r_0)\rho$, substituting these expressions into the last term of $\nabla \mathrm{G}$, we get

$$\nabla G = \frac{1}{4\pi\rho^3}\left[X - X_0 - \left(\frac{r_0}{a}\right)^2 X + X_0\right]$$

on the surface, so that

$$\frac{\partial G}{\partial n} = \frac{X}{a}.\nabla G = \frac{a^2 - r_0^2}{4\pi a\rho^3}.$$

Thus (7.3.1) takes the form

$$u(X_0) = \frac{a^2 - |X_0|^2}{4\pi a}\int\int_{|X|=a}\frac{h(X)}{|X - X_0|^3}\,d\,S.$$

This is the solution to (12). It is the three-dimensional version of the Poisson formula. In more classical notation, it would be written in the usual spherical coordinates as

$$u(r_0, \theta_0, \phi_0) = \frac{a(a^2 - r_0^2)}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{h(\theta, \phi)}{(a^2 + r_0^2 - 2ar_0\cos\psi)^{3/2}}\sin\theta d\theta d\phi,$$

Where ψ denotes the angle between X_0 *and* X.

Notes

In almost the same way, we can use the method of reflection in two dimensions to recover the Poisson formula for

$$u_{xx} + u_{yy} = 0 \ in \ x^2 + y^2 < a^2, \quad u = h \ \ on \ x^2 + y^2 = a^2.\backslash$$

Beginning with the function $(1/2\pi)\log r,$ we find (see Exercise 11) that

$$G(X, X_0) = \frac{1}{2\pi}\log\rho - \frac{1}{2\pi a}\log\left(\frac{r_0}{a}\rho^*\right)$$

and hence that

$$u(X_0) = \frac{a^2 - |x_0|^2}{2\pi a}\int_{|X|=a}\frac{h(x)}{|X - X_0|^2}\,ds,$$

Which is exactly the same as the Poisson formula (6.3.14), which we found earlier in a completely different way?

EXERCISE:

1. Find the one-dimensional Green's function for the interval $(0,l)$. The three properties defining it can be restated as follows.

(i)It solves $G''(x) = 0 \ for \ x \neq x_0$ ("harmonic")

(ii) $G(0) = G(l) = 0.$

(iii) $G(x)$s is continuous at $x_0 \ and \ G(x) + \frac{1}{2}|x - x_0|$ at x_0.

2. Verify directly from (3) or (4) that the solution of the half-space problem satisfies the condition at infinity:

$$u(x) \to 0 \qquad as \ |X| \to \infty.$$

Assume that $h(x, y)$ is a continuous function that vanishes outside some circle.

3. Show directly from (3) that the boundary condition is satisfied: $u(x_0, y_0, z_0) \to h(x_0, y_0) as\, z_0 \to 0.$ Assume $h(x, y)$ is continuous and bounded. [**Hint:** Change variables $s^2 = \left[(x - x_0)^2 + (y - y_0)^2\right] / z_0^2$ and use the fact that

$$\int_0^\infty s\left(s^2 + 1\right)^{-3/2} ds = 1.]$$

4. Verify directly from (3) that the solution has derivatives of all orders in $\{z > 0\}$. Assume that $h(x, y)$ is a continuous function that canishes outside some circle.(Hint: See Section A.3 for differentiation under an integral sign.)

5. Notice that the function x, y is harmonic in the half-plane $\{y > 0\}$ and vanishes on the boundary line $\{y = 0\}$. The function 0 has the same properties. Does this mean that the solution is not unique? Explain.

6. (a) Find the Green's function for the half-plane $\{(x, y) : y > 0\}$.

(b) Use it to solve the Dirichlet problem in the half-plane with boundary values $h(x)$.

(c) Calculate the solution with $u(x, 0) = 1$.

7. (a) If $u(x, y) = f(x / y)$ is a harmonic function, solve the *ODE* satisfied by f.

(b) Show that $\partial u / \partial r \equiv 0$, where $r = \sqrt{x^2 + y^2}$ as usual.

(c) Suppose that $\upsilon(x, y)$ is any function in $\{y > 0\}$ such that $\partial \upsilon / \partial r \equiv 0$. Show that $\upsilon(x, y)$ is a function of the quotient x / y..

(d) Find the boundary values $\lim_{y \to 0} u(x, y) = h(x)$.

Notes

8. (a) Use Exercise 7 to find the harmonic function in the half-plane $\{y>0\}$ with the boundary data $h(x)=1 \text{ for } x>0, h(x)=0 \text{ for } x<0.$

(b) Do the same as part (a) for the boundary data

$$h(x)=1 \text{ for } x>a, h(x)=0 \text{ for } x<a.$$

(Hint: Translate the preceding answer.)

(c) Use part (b) to solve the same problem with the boundary data $h(x)$, where $h(x)$ is any step function. That is,

$$h(x)=c_j \quad \text{for } a_{j-1}<x<a_j \quad \text{for } 1\le j\le n,$$

Where $-\infty=a_0<a_1<\ldots<a_{n-1}<a_n=\infty$ and the c_j are constants.

9. Find the Green's function for the tilted half-space $\{(x,y,z):ax+by+cz>0\}$. (Hint: Either do it from scratch by reflecting across the tilted plane, or change variables in the double integral (3) using a linear transformation.)

10. Verify the formula (11) for $G(X,0)$, the Green's function with its second argument at the centre of the sphere.

11. Find the potential of the electrostatic field due to a point charge licated outside a grounded sphere. (Hint: This is just the Green's function for the exterior of the sphere. Find it by the method of reflection.)

12. Find the Green's function for the half-ball $D=\{x^2+y^2+z^2<a^2, z>0\}$. (Hint: The easiest method is to use the solution for the whole ball and reflect it across the plane.)

13. Do the same for the eighth of a ball

$$D=\{x^2+y^2+z^2<a^2, \mathrm{x}>0, y>0, z>0\}.$$

14. (a) Show that if $\upsilon(x,y)$ is harmonic , so is $u(x,y)=\upsilon\left(x^2-y^2,2xy\right)$.

(b) Show that the transformation $(x,y)\mapsto\left(x^2-y^2,2xy\right)$ maps the first quadrant onto the half-plane $\{y>0\}$. (Hint: Use either polar coordinates or complex variables.)

15. Consider the four-dimensional laplacian $\Delta u=u_{xx}+u_{yy}+u_{zz}+u_{ww}$.

Shown that its fundamental solution is r^{-2}, where $r^2=x^2+y^2+z^2+w^2$.

16. Solve the Neumann problem in the half-plane: $\Delta u=0\ in\ \{y>0\}, \dfrac{\partial u}{\partial y}=h(x) on\ \{y=0\}$ with $u(x,y)$ bounded at infinity. (Hint: Consider the problem satisfied by $\upsilon=\dfrac{\partial u}{\partial y}$.)

17. Solve the Neumann problem in the quarter-plane $\{x>0,y>0\}$.

Check your progress

3. Explain about half space and sphere

7.6 LET US SUM UP

In this unit we have discussed about Green's first identity, Greens's second identity, Green's functions and Half –space and sphere. In this unit the divergence theorem and vector notation will be used extensively. Dirichlet's principle is an important mathematical theorem based on the physical idea of energy. It states among all the functions $w(X)$ in D that satisfy the Dirichlet boundary condition

$w = h(X)$ $on\,bdy\,D$. Green’s second identity is the higher-dimensional version of the identity. It leads to a basic representation formula for harmonic functions.

7.7 KEY WORDS

1. The Green’s identities for the laplacian lead directly to the maximum principle and to Dirichlet’s principle about minimizing the energy.
2. The Green’s function is a kind of universal solution for harmonic functions in a domain.
3. In three dimensions the mean value property states that the average value of any harmonic function over any sphere equals its value at the center.
4. Green’s first identity and Green’s second identity
5. Dirichlet principal states among all the functions $w(X)$ in D that satisfy the Dirichlet boundary condition.

7.8 QUESTIONS FOR REVIEW

1. Discuss about Green’s first identity
2. Discuss about Green’s second identity
3. Discuss about Green’s functions
4. Discuss about Half-space and sphere

7.9 SUGGESTED READINGS AND REFERENCES

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7.Duchateau and D.W. Zachmann, "Partial Differential Equations,"
Schaum, Outline Series, McGraw hill Series.

8.Partial Differential Equations, -Walter A.Strauss

9.Partial Differential Equations,-John K.Hunter

10. Partial Differential Equations,*Erich Mieremann*

11. Partial Differential Equations,-*Victor Ivrii*

7.10 ANSWERS TO CHECK YOUR PROGRESS

1. See section 7.3
2. See section 7.3
3. see section 7.6